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Nonlinear Transformation of Acoustic Waves in Microinhomogeneous Media with Relaxation

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Summary
A nonlinear equation of state and the corresponding wave equation are derived in the framework of a rheological model suggested for elastic media containing defects with relaxation. The defects in the model are considered as highly compliant visco-elastic inclusions that exhibit a nonlinear stress-strain dependence. For real solids, these inclusions may model, for example, cracks, intergrain contacts and other similar defects. The proposed model thus can be applied to a wide class of microinhomogeneous materials (Earth rocks, engineering materials, damaged metals, etc.), which are also called "mesoscopic" solids. The derived equations consistently comprise the following material properties connected to the presence of the defects: (i) the microstructure-induced absorption (including near-constant Q-factor), (ii) the complementary dispersion of sound velocity, (iii) increased magnitude of the "mesoscopic" elastic nonlinearity and (iv) frequency dependence of the nonlinearity. To illustrate the aforementioned manifestations of the material microstructure, a few basic nonlinear effects (second and difference-frequency harmonics generation, and self-demodulation of high-frequency pulses) are analyzed in the framework of the derived equations. Main distinctions of the effects compared to the case of "classical" lattice (atomic) nonlinearity of homogeneous solids are pointed out.

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1. Introduction

The wide class of the so-called microinhomogeneous [1, 2, 3] (alternatively named "mesoscopic" [4]) media is a subject of intensive studies in acoustics and seismics for the last decades. This class comprises a large number of solids which contain, for example, cracks, inter-grain contacts, dislocations and similar defects whose characteristic size is larger than the atomic scale and small compared to the acoustic wave length. It is well known that solids with such microstructure (for example, almost all Earth rocks) exhibit important acoustic features that significantly differentiate them from monocrystals and homogeneous amorphous solids [1, 2, 3, 4, 5, 6]. For example, for quite a long time it was known that many rocks in a wide frequency band exhibit near-constant Q-factor together with logarithmic dispersion of elastic wave velocity [5, 6]. In recent years, a large amount of data was obtained which indicate that the same class of microinhomogeneous media also exhibits very pronounced nonlinear elastic properties in contrast to weakly nonlinear homogeneous solids with lattice nonlinearity [2, 3, 4, 7, 8, 9]. This difference in nonlinear properties is not only qualitative, but in many cases quantitative. In particular, there is evidence that nonlinearity in microinhomogeneous solids may exhibit pronounced frequency dependence [10, 11, 12]. To adequately comprise the onset of these properties, the equation of state of such media, often being non-analytical [13], has to meet the type and amount of the defects in the material.

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The aforementioned linear and nonlinear manifestations of the microstructure in most pertinent publications, until recently were studied independently. In theory, the relevant models were mostly developed at the phenomenological level [14, 15, 16, 17, 18, 19, 20, 21], which did not elucidate the connection of real structural features of the material to its acoustical linear and nonlinear properties. In the recent series of papers [22, 23, 24, 25, 26], the present authors considered the onset of linear and nonlinear acoustic properties typical of microinhomogeneous media on the basis of a proposed material model in the form of a chain of elastic elements with a small amount of highly compliant visco- and nonlinear-elastic inclusions (defects). Due to the high relative compliance of the inclusions, their strain and the stain-rate are strongly increased compared to the strain and the strain-rate in the surrounding relatively rigid material. As a result, both the dissipation of elastic energy and the deviation from the linear Hooke's law are localized at these highly compliant inclusions. Such localization makes it possible to consider the main part of the material as ideally elastic and linear, whereas the relaxation (that is dissipative and dispersive properties) and the nonlinearity can be taken into account only at the highly compliant inclusions. This approach was recently further developed in papers [27, 28] to derive a dynamic nonlinear equation of state for such media, which takes into account both the structure-induced linear dissipation and dispersion, and the frequency dependence of the material nonlinear elasticity. The latter property is shown to be an inherent feature of microinhomogeneous solids [27, 28], which is closely related to the frequency dependence of linear dissipation (the problem of physical origin of near-constant Q-factor) and the complementary dispersion of the sound velocity.

In the present paper, on the basis of the dynamic equation of state obtained in [27, 28] for microinhomogeneous solids, a nonlinear wave equation is obtained and used for studying some basic nonlinear effects. The main features of the derived equation are pointed out in comparison with conventional nonlinear wave equations for acoustic waves in homogeneous media. It is shown that the inherent dispersion of nonlinearity may essentially impact the nonlinear transformation of elastic waves in such media. For the effective nonlinear parameters, their meaning and frequency behavior essentially depend on the particular type of nonlinear transformation (for example, whether up- or down-conversion in frequency is considered). In the next sections, first, the wave equation will be derived for a rather general type of defect's nonlinearity, then the mentioned features will be discussed in detail considering a few basic nonlinear effects for the instructive case of quadratic-in-strain nonlinearity.

2. Derivation of the nonlinear wave equation

The rheological model of the medium mentioned in the Introduction is shown in Figure 1 and consists of an inhomogeneous chain of masses, linear elastic and nonlinear visco-elastic elements. In this model, homogeneous parts of the chain consisting of relatively rigid elements with the stiffness coefficient κ correspond to the regions of defect-free ideal-elastic medium. Nonlinear visco-elastic elements (whose linear stiffness $\kappa_1 \ll \kappa$) correspond to the soft defects- inclusions. The equations of state of the linear-elastic elements and the visco-elastic nonlinear inclusions, respectively, have the forms:

$$\sigma = E\varepsilon_0, \tag{1}$$

$$\sigma = \zeta E\big[\varepsilon_1 - F(\varepsilon_1)\big] + g\dot{\varepsilon}_1, \tag{2}$$

where $E = \kappa l$ is the elastic modulus; ε_0 and ε_1 are the strains of the rigid elements and soft defects respectively; $\dot{\varepsilon}_1 \equiv \mathrm{d}\varepsilon_1/\mathrm{d}t$ is the rate of the strain; g is the viscous coefficient; ζ is the non-dimensional (relative) coefficient of the defects' compliance ($\zeta \ll 1$); $F(\varepsilon_1)$ is a small nonlinear correction to the constitutive equation of the soft element ($|F(\varepsilon_1)| \ll |\varepsilon_1|$). Note that this function may be non-analytical [13]. If the linear (per unit length of the chain) concentration of the defects equals ν, the strain ε of the medium is related to partial strains ε_0 and ε_1 via the expression:

$$\varepsilon = (1-\nu)\varepsilon_0 + \nu\varepsilon_1. \tag{3}$$

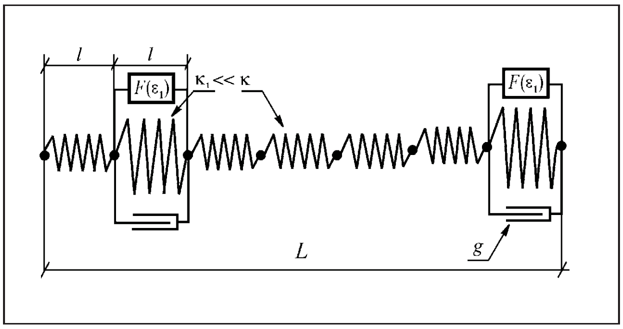

Figure 1. Rheological model of a microinhomogeneous medium.

Using the method of successive approximations, one may obtain from equations (1)–(3) the following equation of state of the medium [27, 28]:

$$\begin{aligned}\varepsilon(\sigma) = \frac{1}{E}\Bigg(&(1-\nu)\sigma + \nu\Omega\int_{-\infty}^{t}\sigma(t_1)\mathrm{e}^{-\zeta\Omega(t-t_1)}\mathrm{d}t_1\Bigg)\\ &+ \nu\Omega\zeta\int_{-\infty}^{t}\mathrm{e}^{-\zeta\Omega(t-t_1)}\\ &\cdot F\left(\frac{\Omega}{E}\int_{-\infty}^{t_1}\sigma(t_2)\mathrm{e}^{-\zeta\Omega(t_1-t_2)}\mathrm{d}t_2\right)\mathrm{d}t_1,\end{aligned} \tag{4}$$

where $\Omega = E/g$, so that $\Omega\zeta$ is the relaxation frequency of the defect.

This equation is valid in the whole range of the defect concentration ($0 \le \nu \le 1$). At the limit $\nu = 0$ we get an ideally elastic linear medium, and at $\nu = 1$, a nonlinear elastic medium with dissipation typical for liquids, gases and homogeneous solids.

In the case of small concentration of the defects, $\nu \ll 1$, equation(4) may be resolved to the "canonical" form $\sigma = \sigma(\varepsilon)$:

$$\begin{aligned}\sigma(\varepsilon) = E\Bigg(&\varepsilon - \nu\Omega\int_{-\infty}^{t}\varepsilon(\tau)\mathrm{e}^{-\zeta_i\Omega(t-t_1)}\mathrm{d}t_1\\ &- \nu\Omega\zeta\int_{-\infty}^{t}\mathrm{e}^{-\zeta_i\Omega(t-t_1)}\\ &\cdot F\left(\Omega\int_{-\infty}^{t_1}\varepsilon(t_2)\mathrm{e}^{-\zeta_i\Omega(t_1-t_2)}\mathrm{d}t_2\right)\mathrm{d}t_1\Bigg).\end{aligned} \tag{5}$$

Note that equations (4), (5) are written for the case of identical defects. The concentration ν of real (not identical) defects should be characterized by a distribution function $\nu = \nu(\zeta, \Omega)$, and contributions from different defects should be summed over this distribution. For convenience, let us introduce the following notation for the relaxation operator and for the integral operators corresponding to the summation over defect parameters:

$$R[\ldots] = \Omega\int_{-\infty}^{+\infty}[\ldots]H(t-t_1)\mathrm{e}^{-\zeta\Omega(t-t_1)}\mathrm{d}t_1, \tag{6}$$

$$I[\ldots] = \iint[\ldots]\nu(\Omega,\zeta)\,\mathrm{d}\Omega\,\mathrm{d}\zeta, \tag{7}$$

$$J[\ldots] = \iint [\ldots]\nu(\Omega,\zeta)\zeta \, d\Omega \, d\zeta, \quad (8)$$

where $H(t)$ is the Heaviside step-function. Note that below, for convenience, we shall mostly limit the dependence of the defect distribution to one parameter, the compliance, corresponding to the choice $\nu = \nu(\zeta)$. When necessary, the complete form of the operators in equations (7), (8), with integration over both variables ζ and Ω may be readily retained. In terms of the introduced notation, the equation of state in equation (5) for the case of different defects may be rewritten in the following compact form:

$$\sigma(\varepsilon) = E\Big\{\varepsilon - I\big[R(\varepsilon)\big] - J\big[R\big(F[R(\varepsilon)]\big)\big]\Big\}. \quad (9)$$

Substituting equation (9) into equation of motion ($\rho U_{tt} = \sigma_x, \varepsilon = U_x$, where ρ is density) we get the following wave equation for displacement $U(x,t)$:

$$U_{tt} - C^2 U_{xx} = -C^2\Big\{I\big[R(U_{xx})\big] + J\big[R\big(F_x[R(U_x)]\big)\big]\Big\}, \quad (10)$$

where $C = \sqrt{E/\rho}$ is the wave velocity in the defect-less chain and corresponds to the high-frequency limit of the wave in the chain with defects. Note that in the derivation of equation (10), "geometrical" nonlinearity is neglected in comparison with the material nonlinearity, that is the contribution of the nonlinearity of equation of state, equation (9), connected to the presence of the defects. In this approximation, ρ =const., $\varepsilon = U_x$.

Further, using the method of slowly varying profile, in terms of the retarded time $\tau = t - x/C$, equation (10) is transformed into the equation for the slowly varying velocity $V = U_\tau$:

$$V_x + \frac{1}{2C}I\big[R(V_\tau)\big] + \frac{1}{2C}J\Big\{R\big[F_\tau\big(-R(V)/C\big)\big]\Big\} = 0. \quad (11)$$

Equations (10), (11) obtained for the microinhomogeneous medium with relaxation thus take into account linear attenuation and dispersion, the structure-induced nonlinearity and its frequency dependence due to relaxation at the defects. Though the parameters of the defects were introduced in the equation of state (2) in a phenomenological manner, the consequent derivation of the wave equation is consistently straightforward and does not imply additional phenomenological assumptions. In particular, such a rigorous derivation shows that in the nonlinear term, relaxation at the defects manifests itself in a two-fold manner. First, the relaxation influences the linear response of the medium, which is the argument of the nonlinearity function $F(..)$, this statement being universal for any nonlinear process and arbitrary type of function $F(..)$. Second, together with the argument of the nonlinearity function, relaxation affects the "output" of the nonlinearity (see the relaxation operator outside the nonlinear function $F(..)$). The resultant effect of relaxation on the nonlinear response thus essentially depends on the particular type of nonlinear transformation (output of the nonlinearity function). As it is shown below, a significant difference exists between the processes with down- or up-conversion in the wave frequency, so that the resultant frequency dependencies of nonlinearity are essentially different, and the material response for different processes cannot be characterized by the same nonlinear parameter(s) in contrast to homogeneous media without relaxation. Therefore, for microinhomogeneous media, the derived evolution equation (11) consistently takes into account the effect of relaxation at the defects on material linear and nonlinear properties, thus being a generalization of the KdV-Burgers equation conventionally used in nonlinear acoustics of homogeneous media [29].

For the particular case of quadratic nonlinearity of the defects ($F(\varepsilon) = \gamma\varepsilon^2$), equation (11) yields:

$$V_x + \frac{1}{2C}I\big[R(V_\tau)\big] - \frac{\gamma}{C^2}J\Big\{R\big[R(V)R(V_\tau)\big]\Big\} = 0. \quad (12)$$

When relaxation in the nonlinear term in equation (12) is neglected and the inclusions are identical, the equation reduces to the earlier considered [29] form with a constant quadratic nonlinear parameter and a linear relaxation term. The derived equations (10)–(12) in particular, allow for description of wave propagation in media with near-constant Q-factor and the corresponding logarithmic dispersion (due to the relaxation-band spectrum), which was discussed in detail in paper [26] in the linear approximation. In the next sections, we shall consider examples of manifestations of the nonlinear-dispersive properties of such media. Namely, second and difference-frequency harmonic generation and self-demodulation of pulses with a high frequency carrier in media with frequency-dependent nonlinearity will be analyzed.

3. Second harmonic generation

Consider the nonlinear process of propagation of elastic harmonic waves in the framework of the evolution equation (12) using the perturbation approach:

$$V(x,\tau) = V_1(x,\tau) + V_2(x,\tau). \quad (13)$$

Here $V_1(x,\tau)$ corresponds to the primary wave, and $V_2(x,\tau)$ describes the nonlinearity-induced correction, $|V_2(x,\tau)| \ll |V_1(x,\tau)|$. In the first approximation, equations (12), (13) yield the next equation for the primary wave:

$$V_{1x} + \frac{1}{2C}I\big[R(V_{1\tau})\big] = 0. \quad (14)$$

Substituting in equation (14) the solution in the form $V_1(x,\tau) = \frac{1}{2}A_1(x)\mathrm{e}^{\mathrm{i}\omega t} + c.c.$, we obtain for the amplitude A_1 of the fundamental harmonic the equation:

$$A_{1x} + \mathrm{i}K_1A_1 = 0, \tag{15}$$

where dispersion correction K_1 is the difference between the current wave number $k = k(\omega)$ corresponding to frequency ω, and the value ω/C corresponding to the high-frequency limit C of the wave velocity:

$$K_1 = k(\omega) - \omega/C = \frac{1}{2C}I\left(\frac{\omega\Omega}{\mathrm{i}\omega + \zeta\Omega}\right). \tag{16}$$

In the full form via the distribution $\nu = \nu(\zeta)$ over compliance of the defects, equation (16) has the form:

$$K_1 = \frac{\omega}{2C}\int_0^1 \left[\frac{\zeta}{\zeta^2 + (\omega/\Omega)^2} - \mathrm{i}\frac{\omega/\Omega}{\zeta^2 + (\omega/\Omega)^2}\right]\nu(\zeta)\,\mathrm{d}\zeta. \tag{17}$$

Note that in case of a wide and smooth distribution $\nu = \nu(\zeta)$, equation (17) yields a wide frequency range in which the medium exhibits near-constant Q and a weak logarithmic dispersion [25, 26].

For boundary condition

$$V(x=0,t) = (A_0/2)\exp(\mathrm{i}\omega t) + c.c.,$$

the first-order equation (15) has a solution $A_1(x) = A_0\mathrm{e}^{-\mathrm{i}K_1x}$. In the second order approximation, equation (12) yields an inhomogeneous equation for the nonlinear correction $V_2(x,t)$:

$$V_{2x} + \frac{1}{2C}I\big[R(V_{2\tau})\big] = \frac{\gamma}{C^2}J\Big\{R\big[R(V_1)R(V_{1\tau})\big]\Big\}. \tag{18}$$

A solution of equation (18) for the second harmonic may be found in the form $V_2(x,\tau) = \frac{1}{2}A_2(x)\mathrm{e}^{2\mathrm{i}\omega\tau} + c.c.$. Then equation (18) yields the equation for the complex amplitude $A_2(x)$:

$$A_{2x} + \frac{\mathrm{i}}{2C}I\left(\frac{2\omega\Omega}{\zeta\Omega + 2\mathrm{i}\omega}A_2\right) = \frac{\gamma A_0^2}{2C^2}J\left(\frac{\mathrm{i}\omega\Omega^3}{(\zeta\Omega + 2\mathrm{i}\omega)(\zeta\Omega + \mathrm{i}\omega)^2}\right)\mathrm{e}^{-2K_1x}. \tag{19}$$

Integrating equation (19) we get an expression for $A_2(x)$:

$$A_2(x) = \frac{\gamma A_0^2}{2C^2}J\left(\frac{\mathrm{i}\omega\Omega^3}{(\zeta\Omega + 2\mathrm{i}\omega)(\zeta\Omega + \mathrm{i}\omega)^2}\right) \cdot \left\{\frac{1 - \mathrm{e}^{\mathrm{i}(K_2 - 2K_1)x}}{K_2 - 2K_1}\right\}\mathrm{e}^{-\mathrm{i}K_2x}, \tag{20}$$

where $K_2 = k(2\omega) - 2\omega/C$ is the dispersion correction to the wave number of the second harmonic, $K_2 = I[\mathrm{i}2\omega\Omega/(\zeta\Omega + 2\mathrm{i}\omega)]/(2C)$, or in the full form:

$$K_2 = \frac{\omega}{C}\int_0^1 \left[\frac{\zeta}{\zeta^2 + (2\omega/\Omega)^2} - \mathrm{i}\frac{2\omega/\Omega}{\zeta^2 + (2\omega/\Omega)^2}\right]\nu(\zeta)\,\mathrm{d}\zeta. \tag{21}$$

Using equations (16) and (21), we find

$$\Re e\{K_2 - 2K_1\}/\Im m\{K_1\} \ll \pi. \tag{22}$$

Here $1/2\Im m\{K_1\}$ is the effective interaction length, and $\Re e\{K_2 - 2K_1\}$ is the dispersion mismatch between the wave numbers of the interacting fundamental and second harmonics. Therefore, condition (22) means that the phase-mismatch between the fundamental and second harmonics within the effective interaction length $L \approx 1/2\Im m\{K_1\}$ is not important, so equation (20) may be simplified as follows:

$$A_2(x) \approx -\frac{\gamma A_0^2}{2C^2}J\left(\frac{\mathrm{i}\omega\Omega^3}{(\zeta\Omega + 2\mathrm{i}\omega)(\zeta\Omega + \mathrm{i}\omega)^2}\right)x\mathrm{e}^{-\mathrm{i}K_2x}. \tag{23}$$

Note that operator $J[..]$ in equation (23) displays the frequency dependence of the elastic nonlinearity for the process of the second harmonic generation, whereas the spatial factor $x\exp(-\mathrm{i}K_2x)$ in (23) is the same as in a "common" medium without dispersion of its nonlinear properties. The structure of the denominator in the argument of operator $J[..]$ in equation (20), (23) shows the above-mentioned two-fold influence of the defects' relaxation both at the fundamental frequency ω and at the second harmonic 2ω. Let us discuss the frequency-dependence of the effective nonlinear parameter starting from the case of identical defects with compliance parameter ζ and density ν, and at small distance ($\exp(-\Im mK_1x) \approx 1$). In this case, for the amplitude of the second harmonic A_2 and its phase, equation (23) yields:

$$|A_2(x)| \approx \frac{\nu\gamma A_0^2}{2C^2}\frac{\omega x\exp(-\Im mK_2x)}{2\zeta^2[1 + (\omega/\zeta\Omega)^2]\sqrt{1 + (2\omega/\zeta\Omega)^2}}, \tag{24}$$

$$\varphi_2 = \arctan\left(\frac{\Re eA_2}{\Im mA_2}\right) = \arctan\left(\frac{2(\omega/\zeta\Omega)[(\omega/\zeta\Omega)^2 - 2]}{5(\omega/\zeta\Omega)^2 - 1}\right). \tag{25}$$

In equation (24) for the second harmonic amplitude, a frequency-dependent non-dimensional factor may be singled out:

$$N_2 = \frac{1}{[1 + (\omega/\zeta\Omega)^2]\sqrt{1 + (2\omega/\zeta\Omega)^2}}, \tag{26}$$

that reduces to unity at $\omega(\zeta\Omega \ll 1$, when relaxation does not affect the nonlinear source in the right-hand side of equation (18). Factor $N_2 \leq 1$ characterizes the ratio of the magnitude of the effective nonlinear parameter in the medium with relaxation to the magnitude ν/ζ^2 of the frequency-independent nonlinear parameter in the absence of relaxation. Note that due to the high compliance of the defects ($\zeta \ll 1$), the non-relaxational nonlinear parameter ν/ζ^2 of the microinhomogeneous material may be much greater than the nonlinear parameter

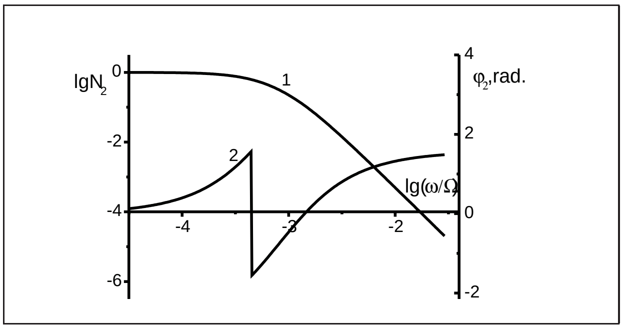

Figure 2. Frequency dependencies of the normalised parameter of nonlinearity N_2 (curve 1, left Y-axis) and phase φ_2 (curve 2, right Y-axis) of the second harmonic in a medium with identical defects ($\zeta = 10^{-3}$).

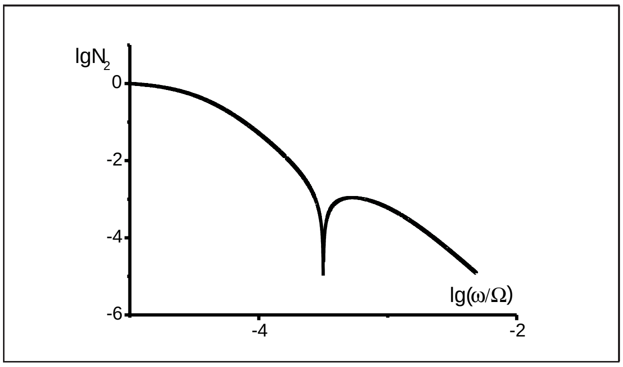

Figure 3. Non-monotonous frequency dependence of the normalised parameter of nonlinearity N_2 for the second harmonic generation in a medium with two types of the defects ($\zeta_1 = 10^{-3}$, $\zeta_1/\zeta_2 = 20$).

γ of the defects [22, 23, 24]. In the structure of the relaxed, frequency-dependent parameter N_2, the presence of the factor $1/\sqrt{1+(2\omega/\zeta\Omega)^2}$ corresponds to the relaxational response of the medium at the nonlinearly generated second harmonic, whereas the factor $1/[1+(\omega/\zeta\Omega)^2]$ is connected to the squared relaxation response at the fundamental frequency. The frequency dependencies of N_2 and φ_2 are shown in Figure 2. The curves exhibit a rather rapid decrease of the parameter $N_2 \sim \omega^{-3}$, when the frequency ω exceeds the defect relaxation frequency $\zeta\Omega$, and a rapid phase variation by π radians around frequency $\omega = \zeta\Omega/\sqrt{5}$. Such a phase behavior may lead to another interesting effect when defects with different relaxation frequencies exist in the material. Indeed, the second harmonic components produced by different defects may superimpose either in-phase (constructively) at one frequency or out of phase (destructively) at another frequency, thus resulting in a non-monotonous frequency dependence of parameter N_2. An example of such a dependence is given in Figure 3 for the case of two types of defects with equal concentrations and different compliance parameters related as $\zeta_2/\zeta_1 = 20$.

For natural media (e.g., rocks), a wide distribution of defect parameters is more realistic than a narrow or bimodal distribution. If such a wide distribution is approxi-

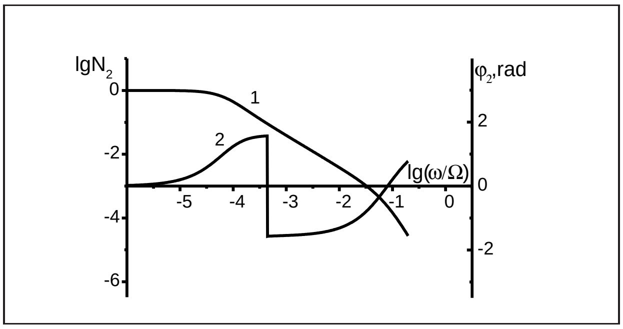

Figure 4. Frequency dependencies of the normalised parameter of nonlinearity N_2 (curve 1, left Y-axis) and phase φ_2 (curve 2, right Y-axis) of the second harmonic in a medium with defects characterised by a wide distribution in the elasticity parameter ζ (the distribution boundaries are $a = 10^{-3}$, $b = 10^{-1}$).

mated, for example, by a wide Π-shape function

$$\nu(\zeta) = \nu_0 \quad \text{at} \quad \zeta \in [a,b], \quad \text{and}$$
$$\nu(\zeta) = 0 \quad \text{at} \quad \zeta \notin [a,b], \quad a \ll b, \tag{27}$$

equation (23) for the complex amplitude A_2 of the second harmonic takes the following form:

$$A_2(x) \approx -\frac{\gamma\omega A_0^2}{2C^2}\Bigg[\Bigg\{\frac{2\Omega}{\omega}\Big[\arctan(\zeta\Omega/2\omega) - \arctan(\zeta\Omega/2\omega)\Big] + \frac{1}{\zeta[1+(\omega/\zeta\Omega)^2]}\Bigg\}\Bigg|_{\zeta=a}^{\zeta=b} + \mathrm{i}\Bigg\{\frac{\Omega}{\omega}\ln\left[\frac{1+(\omega/\zeta\Omega)^2}{1+(2\omega/\zeta\Omega)^2}\right] + \frac{\omega/\zeta\Omega}{\zeta[1+(\omega/\zeta\Omega)^2]}\Bigg\}\Bigg|_{\zeta=a}^{\zeta=b}\Bigg] x\mathrm{e}^{-\mathrm{i}K_2 x}. \tag{28}$$

In this expression, the frequency behavior of the material nonlinearity is described by the term in the big square brackets. The frequency dependence of the corresponding normalized nonlinear parameter N_2 and phase $\varphi_2 = \arctan(\Re e A_2/\Im m A_2)$ are shown in Figure 4. In the figure, at low frequency $\omega < a\Omega$, the parameter $N_2 = 1$; then in the band $a\Omega < \omega < b\Omega$ the nonlinearity parameter decreases as $N_2 \sim 1/\omega$, and at higher frequencies $\omega > b\Omega$, the parameter $N_2 \sim 1/\omega^3$ as in the discussed above case of identical defects at $\omega > \zeta\Omega$.

4. Generation of the difference-frequency wave

Let us consider now the generation of the difference-frequency wave for bi-harmonic excitation of the medium at frequencies ω_1 and ω_2 and initial amplitudes A_1 and A_2. In this case, the solution of equation (18) is found in the form $V_d(x,\tau) = \frac{1}{2}A_d(x)\exp(\mathrm{i}\omega_d\tau) + c.c.$, where $\omega_d = \omega_1 - \omega_2$. Then the solution of equation (18) for zero boundary condition $A_d(x=0)=0$ in the case of identical defects has the following form:

$$\begin{aligned} A_d(x) &= \frac{\gamma A_1 A_2 \omega_d}{2C^2} \\ &\cdot J\left[\frac{\mathrm{i}\Omega^3}{(\zeta\Omega+\mathrm{i}\Omega_d)(\zeta\Omega+\mathrm{i}\Omega_1)(\zeta\Omega-\mathrm{i}\Omega_2)}\right] \\ &\cdot \left\{\frac{1-\mathrm{e}^{\mathrm{i}(K_d+K_2-K_1)x}}{K_d+K_2-K_1}\right\}\mathrm{e}^{-\mathrm{i}K_d x}, \end{aligned} \quad (29)$$

where the dispersion corrections to the wavenumbers of the primary waves, K_1 and K_2, and the difference-frequency wave, K_d, are given by expressions similar to equations (16), (17) for $K(\omega)$, in which frequency ω should be substituted for the frequency $\omega_{1,2}$ and ω_d, respectively. Unlike the case of second harmonic generation, the phase mismatch between the interacting primary waves and the secondary difference frequency wave may be non-negligible. In the case of identical defects, at strong enough separation in the frequency domain ($\omega_d \ll \zeta\Omega$, $\omega_{1,2} \gg \zeta\Omega$) the phase mismatch $\Re e(K_d+K_2-K_1)x$, generally speaking, may significantly exceed π within the characteristic interaction length $1/\Im m(2K_{1,2})$. However, at small distances (when $|K_1-K_2-K_d|x \ll 1$) the factor in curly brackets in equation (29) may be simplified so that the amplitude and phase of the difference frequency wave are given by the expressions:

$$\begin{aligned} |A_d(x)| &\approx \frac{\nu_0\gamma\omega_d A_1 A_2 x}{2C^2\zeta^2}\Big[\left(1+(\omega_1/\zeta\Omega)^2\right) \\ &\cdot \left(1+(\omega_2/\zeta\Omega)^2\right)\left(1+(\omega_d/\zeta\Omega)^2\right)\Big]^{-1/2}, \end{aligned} \quad (30)$$

$$\begin{aligned} \varphi_d &= \arctan\left(\frac{\Re e A_d}{\Im m A_d}\right) \\ &= \arctan\left(\frac{(\omega_d/\zeta\Omega)[2+\omega_1\omega_2/(\zeta\Omega)^2]}{1+(\omega_1\omega_2-\omega_d^2)/(\zeta\Omega)^2}\right). \end{aligned} \quad (31)$$

As in equation (24) for the second harmonic the amplitude, frequency-dependence of the denominator in equation (30) is determined by the operator $J[..]$ in equation (29) and characterizes the frequency behavior of the effective nonlinearity in the microinhomogeneous material for the case of the difference-frequency signal generation. Note that in the non-relaxed limit ($\Omega \to \infty$), the nonlinear parameter in equation (30) is the same as the non-relaxed parameter $\nu\gamma/\zeta^2$ for the second harmonic (see the discussion of equation 24). However, frequency-dependent parameters for the second and difference-frequency harmonics essentially differ. By analogy with equation (26), it is convenient to introduce the corresponding normalized nonlinear coefficient for the difference-frequency harmonic:

$$\begin{aligned} N_d &= \Big[\left(1+(\omega_1/\zeta\Omega)^2\right)\left(1+(\omega_2/\zeta\Omega)^2\right) \\ &\cdot \left(1+(\omega_d/\zeta\Omega)^2\right)\Big]^{-1/2}. \end{aligned} \quad (32)$$

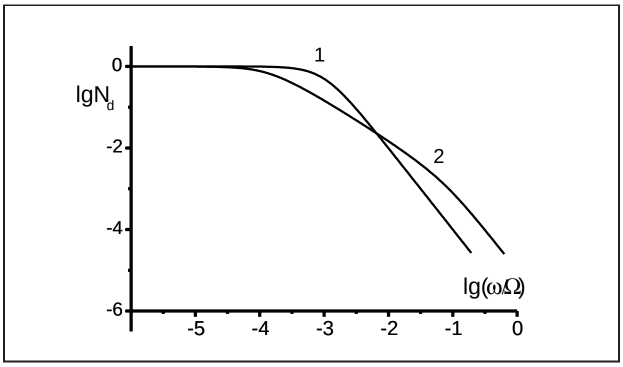

Figure 5. The magnitude of the normalised nonlinearity parameter N_d for the difference-frequency harmonic at fixed $\omega_d = 10^{-5}\Omega$ plotted against high frequency $\omega \approx \omega_{1,2}$. Curve 1 is for the medium with identical defects ($\zeta = 10^{-3}$); curve 2 is for the medium with defects characterised by a distribution in the elasticity parameter ζ (the distribution boundaries $a = 10^{-4}$, $b = 10^{-1}$).

At low frequencies $\omega_1, \omega_2, \omega_d \ll \zeta\Omega$, parameter $N_d \to 1$, which corresponds to the medium without relaxation. In the high-frequency limit ($\omega_1, \omega_2, \omega_d \gg \zeta\Omega$), the normalized parameter N_d decreases as $N_d \propto (\omega_1\omega_2\omega_d)^{-1}$. In the case of close primary frequencies $\omega_1 \approx \omega_2 = \omega$ and for $\omega_d \ll \zeta\Omega$, the magnitude N_d is shown in Figure 5 (curve 1) as a function of frequency ω. Concerning the phase φ_d, note that in contrast to abrupt variations for the second harmonic (see equation 25), equation (31) indicates that the influence of relaxation causes only a smooth variation in the phase φ_d from zero value in the quasistatic ($\omega_1, \omega_2, \omega_d \ll \zeta\Omega$) limit up to $\pi/2$ in the high-frequency limit ($\omega_1, \omega_2, \omega_d \gg \zeta\Omega$). Indeed, the phase of the difference-frequency wave is determined by the difference in phases of the primary waves, whereas the phase of the sum-harmonic (the second one in the case of $\omega_1 = \omega_2 = \omega_d$) is equal to the sum of phases of the primary waves. Therefore, in contrast to the second harmonic, destructive interference of the nonlinear responses of defects with different parameters of compliance does not occur for the difference-frequency component.

In the case of the wide distribution equation (27) of the defects over the compliance parameter ζ, that is, for the relaxation-band medium, the complex amplitude A_d may be written in the following form (here we again assume that $\omega_1 \approx \omega_2 = \omega$, and $\omega_d \ll \omega$):

$$\begin{aligned} \Re e A_d(x) &\approx \frac{\nu_0\gamma A_1 A_2 \omega_d}{2C}\left(\frac{\Omega}{\omega}\right)^2 x\left[\frac{\omega_d}{\Omega}\arctan\left(\frac{\zeta\Omega}{\omega_d}\right)\right. \\ &\left.\left. - \frac{\omega}{\Omega}\arctan\left(\frac{\zeta\Omega}{\omega}\right)\right]\right|_{\zeta=a}^{\zeta=b} \end{aligned} \quad (33)$$

$$\Im m A_d(x) \approx \frac{\nu_0\gamma A_1 A_2}{2C}\frac{\omega_d^2\Omega}{\omega^2}x\ln\left[\frac{1+(\omega/\zeta\Omega)^2}{1+(\omega_d/\zeta\Omega)^2}\right]\Bigg|_{\zeta=a}^{\zeta=b}.$$

These expressions again display the influence of relaxation on the nonlinear transformation via both the high frequency ($\omega_1 \approx \omega_2 = \omega$) primary waves and the secondary

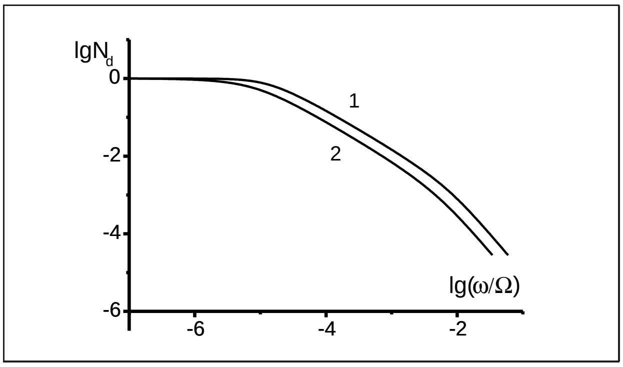

Figure 6. Magnitudes of the normalized parameter of nonlinearity N_d for the difference-frequency harmonic plotted against frequency ω in the case of a medium with defects characterised by a distribution both in the elasticity parameter ζ (the distribution boundaries $a = 10^{-4}$, $b = 10^{-1}$) and in parameter Ω, that is in viscosity. Curve 1 is for $\Omega_b/\Omega_a = 10$; curve 2 is for $\Omega_b/\Omega_a = 10^4$.

signal at difference frequency ω_d. For low difference frequency, $\omega_d \ll a\Omega_2$, the dependence of parameter N_d on the frequency ω is shown in Figure 5 (curve 2). The frequency behavior of the curve corresponds to the response of the medium with relaxation-band spectrum $[a\Omega, b\Omega]$. Namely, at $\omega < a\Omega$, parameter N_d =const., further in the intermediate band $a\Omega < \omega < b\Omega$ the nonlinear parameter decreases approximately as $N_d \sim \omega^{-1}$, and at $\omega > b\Omega$ the rate of the decrease becomes higher, $N_d \sim \omega^{-2}$ as in the case of identical defects at high frequency $\omega > \zeta\Omega$.

Note that the examples considered above related to the case of defects distributed over their parameter of compliance, $\nu = \nu(\zeta)$, whereas a more realistic situation is a distribution over both the compliance and the effective viscosity of the defects, that is $\nu = \nu(\zeta, \Omega)$. Figure 6 illustrates that when the defects have a wide distribution over their compliance parameter ζ, the influence of the additional distribution over Ω does not significantly change the medium response. The curves in Figure 6 show examples of the frequency dependence of effective parameter N_d in cases of a wide Π-shape distribution over compliance ζ and over both parameters Ω and ζ. The curves indicate that for wide distribution over parameter ζ, the difference between the cases of a fixed Ω and quite a wide Π-shape distribution over Ω (with 4 order difference between the boundary values, $\Omega_b/\Omega_a = 10^4$) is not significant (see also the discussion of the distribution over Ω in papers [26, 28]).

5. Self-demodulation of pulses with a high-frequency carrier

Let us consider another example of the manifestation of frequency-dependent nonlinearity in microinhomogeneous solids, namely the effect of self-demodulation of pulses with a high-frequency carrier, which is a well known phenomenon in nonlinear underwater acoustics [30] and is readily observed in microinhomogeneous solids [31]. Unlike the case of frequency-independent nonlinearity of pure water, the example considered below indicates that relaxation properties of the medium affect not only the magnitude of the nonlinear parameter, but also qualitatively (functionally) influence the relation between the shape of the primary wave modulation function and the form of the demodulated pulse.

To describe the nonlinear transformation of the pulse, we shall use equations (14), (18) for the primary and secondary waves. In this section we consider the case of the wide relaxation spectrum, equation (27), which in the linear approximation manifests itself in a near-constant Q of the medium and can model, for example, properties of rocks. For the primary wave with a sinusoidal carrier and slow pulse modulation, the boundary condition may be written in the following form:

$$V(x=0,\tau) = V_0\Phi(\tau/T)\sin\omega_0\tau, \quad \omega_0\tau \gg 1. \qquad (34)$$

We suppose that both the carrier frequency ω_0 and the demodulated pulse spectrum belong to the range $a\Omega < \omega < b\Omega$ in which the phase velocity of the waves exhibits weak dispersion, whereas the Q-factor is nearly constant and the attenuation coefficient nearly linearly depends on frequency. Equations (17), (27) then yield the following correction to the complex wave number:

$$K(\omega) = \frac{\nu_0\omega}{4C}\left\{ \ln\left[\frac{b^2 + (\omega/\Omega)^2}{a^2 + (\omega/\Omega)^2}\right] + \mathrm{i}2\arctan\left[\frac{b\Omega/\omega - a\Omega/\omega}{1 + ba(\Omega/\omega)^2}\right]\right\}. \qquad (35)$$

Therefore, in the frequency band $a\Omega < \omega < b\Omega$ that we consider here,

$$\Re e K(\omega) \approx \frac{\nu_0\omega}{2C}\ln\left[\frac{b}{\omega/\Omega}\right], \qquad (36)$$

$$\Im m K(\omega) \approx \frac{\pi\nu_0\omega}{4C}. \qquad (37)$$

Using equations (35)–(37), by analogy with the estimate given in equation (22), one may readily estimate that for spectral components belonging to the band $a\Omega < \omega < b\Omega$, it is possible to neglect phase mismatch between the interacting waves within the decay length $1/\Im m K(\omega_0)$ of the primary high-frequency wave. Therefore, within the interaction length, the frequency-dependence of the wave velocity may be neglected, and the narrow-band primary wave thus may be approximated by the following expression via the retarded time:

$$V_1(x,\tau) = \frac{V_0}{2}\Phi(\tau/T)\mathrm{e}^{-\chi\omega_0 x + \mathrm{i}\omega_0\tau} + c.c., \qquad (38)$$

where according to equation (37), the coefficient $\chi \approx \pi\nu_0/4C$. In order to find the demodulated pulse, the nonlinear source in the right-hand side of equation (18) should be averaged over a time scale larger than the carrier period $2\pi/\omega_0$ and smaller than the scale T of the modulation function $\Phi(\tau/T)$. In the structure of the low-frequency nonlinear source

$$Q(x,\tau) = \frac{\gamma}{2C^2}\frac{\partial}{\partial\tau}J\Big\{R\Big[\big\langle R^2(V_1)\big\rangle\Big]\Big\}, \qquad (39)$$

(where $\langle..\rangle$ denotes time-averaging), it is possible to neglect slow modulation when evaluating the relaxed primary wave $R(V_1)$. Thus the quadratic nonlinear term $\langle R^2(V_1)\rangle$ in equation (39) can be approximately represented as:

$$\big\langle R^2(V_1)\big\rangle = \frac{\Omega^2 V_0^2 \Phi^2(\tau/T)}{2(\omega_0^2+\zeta^2\Omega^2)}\mathrm{e}^{-2\chi\omega_0 x}. \qquad (40)$$

The relaxation operator outside averaging brackets $\langle..\rangle$ in equation (39) is applied to the low-frequency term (40). Then the linearized equation (18) with the nonlinear low-frequency source for the demodulated pulse may be Fourier-transformed and solved for amplitudes of the pulse Fourier-harmonics as was done above for the second harmonic and the difference frequency wave. The accepted assumption $a\Omega < 2\pi/T < \omega_0 < b\Omega$ simplifies the problem, since we may neglect the mutual phase mismatch between frequency components within the interaction length $L \sim 1/(2\chi\omega_0)$. Further, the temporal shape of the nonlinearly generated pulse may be found by inverse Fourier transformation of the solution found in the frequency domain:

$$V_2(x,\tau) = \frac{\gamma V_0^2}{4C}\mathcal{F}^{-1}\otimes J\Bigg[\frac{\mathrm{i}\omega\Omega^3\mathcal{F}\otimes\Phi^2(\tau/T)}{(\omega_0^2+\zeta^2\Omega^2)(\mathrm{i}\omega+\zeta\Omega)} \cdot \frac{1-\exp[-\chi(2\omega_0-\omega)x]}{\chi(2\omega_0-\omega)}\mathrm{e}^{-\chi\omega x}\Bigg], \qquad (41)$$

where the Fourier operator $\mathcal{F}$ has the meaning $\mathcal{F}\otimes f(\tau) = \int_{-\infty}^{\infty} f(\tau)\exp(-\mathrm{i}\omega\tau)\mathrm{d}\tau$. Equation (41) may be further simplified at $x \ll 1/(2\chi\omega_0)$ and $x \gg 1/(2\chi\omega_0)$. At small distance $x \ll 1/(2\chi\omega_0)$, the factor $(1-\exp[-\chi(2\omega_0-\omega)x])/\chi(2\omega_0-\omega)$ reduces to x, so that $V_2(x,\tau)$ initially grows linearly with distance. At larger distance, $x \gg 1/(2\chi\omega_0)$, beyond the interaction length, the factor $(1-\exp[-\chi(2\omega_0-\omega)x])/\chi(2\omega_0-\omega) \approx 1/\chi(2\omega_0-\omega)$.

As an instructive concrete example we consider the case of the Lorentz type modulation function, $\Phi^2(\tau/T) = 1/[1+(\tau/T)^2]$, with the Fourier transform $\mathcal{F}\otimes\Phi^2(\tau/T) = \pi T\exp(-\omega T)$. In this case, the argument of the operator $\mathcal{F}^{-1}$ in equation (41) may be analytically found via Euler functions, and then the inverse transform $\mathcal{F}^{-1}$ is readily evaluated numerically. Note that the influence of the attenuation $\exp(-\chi\omega x)$ in expression (41) results in a self-similar transformation of the pulse spectrum: $\exp(\omega T) \to \exp[(T+x\chi)\omega]$. Examples of calculated shapes of demodulated pulses in the medium with the wide spectrum, equation (27) of defect compliance are shown in Figure 7 together with the reference pulse corresponding to a homogeneous medium with conventional frequency-independent "instantaneous" nonlinearity. For the latter case of "instantaneous" nonlinearity, the shape of the pulse may be obtained from equation (41) in the limit $\omega \ll \zeta\Omega$. In this case, at small distances, where the pulse spectrum is not affected by linear attenuation, the pulse temporal form is proportional to the first derivative of the squared primary pulse (the symmetrical solid curve 1 in Figure 7):

$$V_2^{\mathrm{hom}}(x,\tau) \propto \frac{\partial}{\partial\tau}\Phi^2(\tau/T). \qquad (42)$$

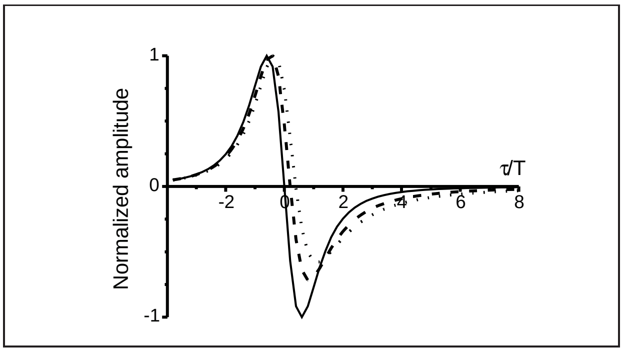

Figure 7. Temporal forms of the demodulated pulses in the relaxation-band medium with frequency-dependent nonlinearity (dashed and dotted lines), and a reference symmetrical pulse in the medium with frequency-independent quadratic nonlinearity (the solid line). The dashed and dotted curves correspond to different positions of the demodulated pulse spectrum within the relaxation band spectrum $[a\Omega, b\Omega]$ of the defects. The dotted curve is for $a = 10^{-6}$, $b = 10^{-2}$; the dashed curve is for $a = 10^{-5}$, $b = 10^{-1}$. The initial inverse duration of the pulse $1/T = 10^{-3}\Omega$ in all cases.

Figure 7 shows that in the medium with relaxation, the pulse shape is distorted (the asymmetrical dashed and dotted curves), the extent of the distortion being dependent on the position of the demodulated pulse spectrum within the frequency band of near-constant Q. The origin of such an asymmetrical distortion may be readily understood from the structure of expression (41). Indeed, in the medium containing defects with a wide relaxation-band spectrum, the resultant pulse shape is formed by contributions of two significantly different fractions of the defects. The first fraction consists of the defects with "instantaneous" reaction, those whose relaxation frequencies $\zeta\Omega$ are much higher than the pulse spectral components, $\zeta\Omega \gg \omega$. The contribution of this fraction to the form of the pulse is proportional to the derivative $\partial\Phi^2(\tau/T)/\partial\tau$ as in the case of an homogeneous medium with "instantaneous" nonlinearity. The second fraction is constituted by defects with low relaxation frequencies $\zeta\Omega \ll \omega$. For these defects, the factor $1/(\mathrm{i}\omega+\zeta\Omega)$ in equation (41) may be approximated by $1/\mathrm{i}\omega$, which corresponds to a spectral representation of the integration over time τ. This fraction thus gives to the resultant shape of the demodulated pulse a contribution close to $\Phi^2(\tau/T)$ instead of the derivative $\partial\Phi^2(\tau/T)/\partial\tau$. Superposition of these contributions with and without the derivative, $\partial\Phi^2(\tau/T)/\partial\tau$ and $\Phi^2(\tau/T)$, produces an asymmetrical pulse shape as displayed in Figure 7 (the dashed and dotted curves). Note that in the case of $\zeta\Omega \gg \omega$ for almost all defects, the second contribution may become dominant, so that the demodulated pulse shape will be close to $\Phi^2(\tau/T)$. However, since $1/(\mathrm{i}\omega + \zeta\Omega) \approx 1/\mathrm{i}\omega$ is only an approximation, the pulse (41) will have a long, small-amplitude negative tail, so that the condition of conservation of pulse area for evolution equations (11), (12) will be satisfied (analogously to the case of the conventional KdV-Burgers equation).

6. Conclusion

The results presented in this paper indicate that nonlinear effects in microinhomogeneous (mesoscopic) media with relaxation may exhibit significant differences from homogeneous materials possessing conventional "instantaneous" (atomic) elastic nonlinearity. The analysis has shown that the relaxation properties of the micro-defects determine the onset of linear dissipation and dispersion characteristics of the material, as well as the frequency-dependence of its nonlinearity. The derived equations consistently take into account all the aforementioned material characteristics. For the class of microinhomogeneous materials, the obtained evolution equation is in fact an analogue and generalization of the KdV-Burgers equation that is conventionally used in nonlinear acoustics of homogeneous media.

In the linear approximation, equations (4), (5) and (10), (12), in particular, allow for taking into account a near-constant Q factor and the complementary logarithmic dispersion corresponding to the presence of a wide relaxation-band spectrum of defects. Such linear properties accompanied by pronounced elastic nonlinearity are typical of a wide class of microstructured solids, for example, rocks normally containing cracks, inter-grain contacts and similar defects. Nowadays, there is general consensus that this microstructure essentially determines material acoustic response, but still there is a lack of theoretical models consistently connecting macroscopic acoustical linear and nonlinear characteristics to the microstructure of the medium. The present paper attempts to make up this deficiency and to take a step beyond a purely phenomenological approach to theoretical modeling of structure-induced nonlinearity.

Among other microstructure-induced acoustic properties, dispersion of nonlinearity is seemingly the least studied. Concerning the specific features discussed above, it is essential to note that the structure of the nonlinear terms in equations (10), (11) clearly exhibits a two-fold effect of the defect's relaxation on the material nonlinearity. First, the relaxation affects the material linear response, that is the argument of the nonlinearity function $F(..)$ (see equations 10, 11). Such a manifestation of relaxation is universal for any type of nonlinear process and arbitrary nonlinearity function $F(..)$. Second, the "output" of nonlinearity $F(..)$ is also affected by the relaxation (see the relaxation operator outside the function $F(..)$ in equations (10), (11). The resultant effect, therefore, essentially depends on the concrete type of the nonlinear process, and may significantly differ depending on whether the process results in down- or up-conversion in the wave frequency. Thus in the case of frequency-dependent nonlinearity, the nonlinear response of the material cannot be characterized by a single (or a few) universal nonlinear parameter(s) in contrast to homogeneous media without relaxation.

Due to the dependence of the nonlinearity parameters on the type of nonlinear transformation, in particular, the ratio of the amplitudes of the second A_2 and difference frequency A_d harmonics in homogeneous and microinhomogeneous media with frequency-dependent nonlinearity is significantly different. Equations (24) and (30) yield that this ratio (at $\omega_{1,2} \approx \omega \gg \omega_d$ and $A_{1,2} = A_d$) in the microinhomogeneous material has the form:

$$\left|\frac{A_d(x)}{A_2(x)}\right| = \frac{\omega_d}{2\omega}\sqrt{\frac{(\zeta\Omega)^2 + (2\omega)^2}{(\zeta\Omega)^2 + \omega_d^2}}. \tag{43}$$

In the homogeneous medium with constant nonlinear parameter (which corresponds to the limit $\omega_d < \omega \ll \zeta\Omega$), the ratio (43) is rather small, $|A_d(x)/A_2(x)| = \omega_d/2\omega \ll 1$, whereas in media with relaxation, the ratio (43) may reach unity at $\omega \gg \zeta\Omega$. Besides, for second harmonic generation, it was shown that in media containing defects with different parameters, the effective nonlinear coefficient may exhibit non-monotonous frequency dependence. In contrast, for difference-frequency generation, such a non-monotonity cannot occur. As another instructive example of the effect of the dispersion of nonlinearity, the distortion of the shape of low-frequency demodulated pulses due to the influence of frequency-dependent nonlinearity was discussed in the previous section.

Therefore, the performed analysis and considered examples have confirmed that, in addition to the conventionally-considered influence via linear dispersion and dissipation, the effect of relaxation via frequency-dependence of nonlinearity may significantly change the character of nonlinear processes.

In this paper, detailed discussions of concrete examples are limited to the case of a quadratic-in-strain nonlinear term. However, wave equations (10), (11) were derived for a more general nonlinearity function and could be readily applied to other types of nonlinearity. In particular, it is generally accepted that equations of state for microinhomogeneous media should incorporate hysteresis [2, 3, 4]. It is known that memory properties of hysteretic materials allow for one-to-one approximation of stress via material strain and its time derivative only in special cases of loading of the material. For example, such a description is possible for a (quasi)periodic stress-strain process with a single maximum and a single minimum over the period. In these cases, the corresponding equation of state of may be approximated, for instance, by a piece-wise power hysteretic [2, 3, 4, 11, 12, 13, 18, 19, 20, 21] or, generally speaking, some other non-analytical function [13]. With a hysteretic nonlinearity in function $F(..)$, equations (4)–(11) provide a description of microinhomogeneous media with defects exhibiting both relaxation and nonlinear-hysteretic properties. The developed theory thus may be applied to the interpretation of experimental data indicating the frequency-dependence of nonlinear properties of hysteretic materials [10, 11, 12]. Note finally, that due to the aforementioned structure-induced nonlinearity and relaxation in microinhomogeneous solids, nonlinear transformations of elastic waves in such media exhibit essential quantitative and qualitative differences from homogeneous media, opening attractive possibilities for diagnostics of material microstructure.

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