Acoustical Physics, Vol. 47, No. 2, 2001, pp. 178–183. Translated from Akusticheskiĭ Zhurnal, Vol. 47, No. 2, 2001, pp. 220–226.

REVIEWS

The Equation of State of a Microinhomogeneous Medium and the Frequency Dependence of Its Elastic Nonlinearity

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Received August 28, 1999

Abstract—In the framework of a rheological model, a nonlinear dynamic equation of state of a microinhomogeneous medium containing nonlinear viscoelastic inclusions is derived. The frequency dependences of the effective nonlinear parameters are determined for the difference frequency and second harmonic generation processes in the case of a quadratic elastic nonlinearity. It is shown that the frequency dependence of the nonlinear elasticity of the medium is governed by the linear relaxation response of the inclusions at the primary excitation frequency, as well as by the relaxation of the inclusions at the nonlinear generation frequencies. *© 2001 MAIK "Nauka/Interperiodica".*

In recent years, the theory of media that exhibit a strong acoustic nonlinearity has been extensively developed. These media include different kinds of rock, some metals, and some structural materials. To date, it has been established that the nonlinear properties of such media are connected with various structural inclusions (or microinclusions) whose dimensions are large relative to the interatomic distances and small relative to the characteristic dimension of acoustic disturbances. In acoustics, such media are referred to as microinhomogeneous media [1–4]. As a rule, the acoustic properties of microinhomogeneous media cannot be described in terms of the classical (five- or nine-constant) theory of elasticity [5]. Firstly, the damping constant of such media is frequency-independent within a sufficiently wide frequency band [6], whereas the damping constant of homogeneous media is a linear function of frequency. Secondly, the elastic nonlinearity of homogeneous media is frequency-independent, whereas the nonlinearity of microinhomogeneous media can depend on frequency [7, 8]. Therefore, an adequate model and a corresponding equation of state should be developed to describe the nonlinear wave processes in microinhomogeneous media. The linear and nonlinear rheological models of a microinhomogeneous medium, which were proposed earlier in [9–14], provide the explanation for the frequency-independent behavior of the Q-factor and the strong elastic nonlinearity observed in such media. This paper combines and extends these models to derive a dynamic nonlinear equation of state of a microinhomogeneous medium and analyzes the frequency dependences of some nonlinear effects in the interaction between elastic disturbances in a medium if this kind.

Consider a rheological model of a nonlinear microinhomogeneous medium. As we noted above, microinhomogeneous media contain various inclusions (grains, cracks, dislocations, etc.) whose characteristic dimensions are small relative to the acoustic wavelength. In most cases, the compressibility of these inclusions is higher than that of the surrounding homogeneous material. Due to the higher compressibility of the inclusions, an elastic stress that occurs in their vicinity creates a higher strain (and, accordingly, strain rate), which is much higher than the average strain (and strain rate) in the medium. Therefore, the dissipation and the elastic nonlinearity of the medium are governed by the effect of these highly compliant inclusions. In order to derive the equation of state of the medium, we consider its part of length L much smaller than the characteristic wavelength λ. In such a region, the strain can be treated as quasi-static, which allows us to ignore the inertial properties of the material. Therefore, the rheological model of the microinhomogeneous medium can be represented by a nonuniform chain of linear elastic and nonlinear viscoelastic elements connected in series, as shown in Fig. 1. In this chain model, the uniform parts consisting of stiff elements (with the elastic coefficient κ) correspond to the inclusion-free regions of a perfectly elastic medium, while the nonlinear viscoelastic elements (with the elastic coefficients $\kappa_i \ll \kappa$) correspond to the compliant inclusions. We assume that the stiff and compliant elements of the chain are of equal length l, so that their number within the length L is equal to N, where $Nl = L$, and the number of inclusions is $N_1 = \nu N$, where the dimensionless coefficient ν is the relative (per-unit-volume) concentration of these inclusions.

Rheological models similar to the model shown in Fig. 1 were proposed in [9–14] for describing the dissipation and the nonlinear elastic properties of microinhomogeneous media. These models explain the frequency independence of the Q-factor of such media on the basis of the assumption that the distribution of elastic parameters of the viscoelastic inclusions is wide with the nonlinear properties of the inclusions being ignored [9–11]. Conversely, the analysis of the nonlinear elasticity of the medium [12–14] ignored the viscosity of the inclusions and allowed for their nonlinearity. Clearly, when the viscosity of the nonlinear inclusions is taken into account, their effective stiffness proves to increase with the frequency of the acoustic disturbance, which results in an increase in the sound velocity at high frequencies, i.e., in the acoustic dispersion. An increase in the stiffness of the inclusions also decreases their strain and, therefore, decreases the nonlinearity of the medium, which means that its nonlinear elasticity becomes frequency-dependent. Thus, the origin of the sound velocity dispersion and that of the frequency dependence of the nonlinear elasticity of the medium are closely related.

To derive the dynamic equation of state of the microinhomogeneous medium, we use the model shown in Fig. 1 to calculate the elongation $X^{(t)}$ of the chain under the action of stress σ as a sum of elongations of the stiff and compliant elements:

$$X^{(t)} = (N - N_1)\varepsilon_0 l + \sum_{i=1}^{N_1} X_i^{(s)}, \quad (1)$$

where $\varepsilon_0 l$ is the elongation of a stiff element, $X_i^{(s)} = \varepsilon_i l$ is the elongation of the ith inclusion, and ε_0 and ε_i are their relative strains. Dividing both sides of Eq. (1) by the length of the element $L = Nl$, we obtain the expression for the average strain ε:

$$\varepsilon = (1 - \nu)\varepsilon_0 + \nu\varepsilon_i. \quad (2)$$

As we noted above, the stiff elements of the chain are perfectly elastic and are described by the equation

$$\sigma = E\varepsilon_0, \quad (3)$$

where $E = \kappa l$ is the elasticity modulus of the medium consisting of the stiff elements. The equation of state of the ith inclusion characterized by the viscosity and elastic nonlinearity has the form

$$\sigma = \varsigma_i E[\varepsilon_i - F(\varepsilon_i)] + g\dot{\varepsilon}_i, \quad (4)$$

where $\dot{\varepsilon}_i \equiv d\varepsilon_i/dt$ is the inclusion strain rate, ς_i is a dimensionless coefficient that characterizes the relative elasticity of the inclusions ($\varsigma_i = E_i/E \ll 1$), and $F(\varepsilon_i)$ is the small elastic nonlinear correction ($|F(\varepsilon_i)| \ll |\varepsilon_i|$).

Equations (2)–(4) can be used to derive the equation of state of the microinhomogeneous medium, i.e., the function $\sigma = \sigma(\varepsilon)$. For the stiff elements, Eq. (3) yields

$$\varepsilon_0 = \sigma/E. \quad (5)$$

Since the nonlinearity is weak, the strain of the inclusions can be found by the successive approximation technique assuming that $\varepsilon_i = \varepsilon_i^{(1)} + \varepsilon_i^{(2)} + \ldots$, where $|\varepsilon_i^{(2)}| \ll |\varepsilon_i^{(1)}|$. In the linear approximation, the solution to Eq. (4) has the form of the relaxation integral:

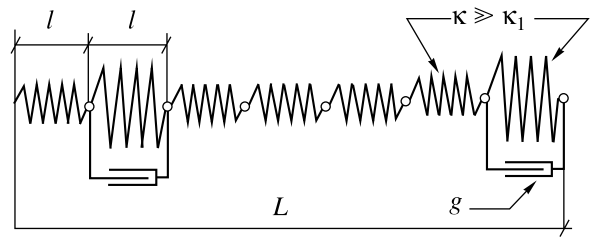

Fig. 1. Rheological model of a microinhomogeneous medium.

$$\varepsilon_i^{(1)}(\sigma) = (\varsigma_i\Omega/E_i)\int_{-\infty}^{t}\sigma(\tau)e^{-\varsigma_i\Omega(t-\tau)}d\tau = (\Omega/E)\int_{-\infty}^{t}\sigma(\tau)e^{-\varsigma_i\Omega(t-\tau)}d\tau, \quad (6)$$

where $\Omega = E/g$ has the frequency dimension, so that $\Omega_i = \varsigma_i\Omega$ is the relaxation frequency of the ith inclusion.

The expression for the nonlinear correction $\varepsilon_i^{(2)}$ has the form

$$\varepsilon_i^{(2)}(\sigma) = \Omega\varsigma_i\int_{-\infty}^{t}e^{-\varsigma_i\Omega(t-\tau)}F\left(\frac{\Omega}{E}\int_{-\infty}^{t}\sigma(\tau)e^{-\varsigma_i\Omega(t-\tau)}d\tau\right)d\tau. \quad (7)$$

Substituting Eqs. (5)–(7) for the strains ε_0 and ε_i into Eq. (2), we obtain the nonlinear dynamic equation of state of the microinhomogeneous medium in the form

$$\varepsilon(\sigma) = \frac{1}{E}\left((1-\nu)\sigma + \nu\Omega\int_{-\infty}^{t}\sigma(\tau)e^{-\varsigma_i\Omega(t-\tau)}d\tau\right) + \nu\Omega\varsigma_i\int_{-\infty}^{t}e^{-\varsigma_i\Omega(t-\tau)}F\left(\frac{\Omega}{E}\int_{-\infty}^{\tau}\sigma(\tau')e^{-\varsigma_i\Omega(t-\tau')}d\tau'\right)d\tau. \quad (8)$$

This equation is valid within the entire range of inclusion concentrations $0 \le \nu \le 1$. The concentrations $\nu = 0$ and $\nu = 1$ (at ς_i = const) correspond to the homogeneous media: at $\nu = 0$, we obtain a perfectly elastic linear medium and, at $\nu = 1$ and ς_i = const, we have a nonlinear elastic medium whose dissipation properties are similar to those of liquids, gases, and homogeneous solids; in this case, the equation of state of the medium coincides with Eq. (4).

When the concentration of the inclusions is small, Eq. (8) can be reduced to the canonical form $\sigma = \sigma(\varepsilon)$:

$$\sigma(\varepsilon) = E\left(\varepsilon - \nu\Omega\int_{-\infty}^{t}\varepsilon(\tau)e^{-\varsigma_i\Omega(t-\tau)}d\tau\right.$$
$$\left.-\nu\,\Omega_i\varsigma\int_{-\infty}^{t} e^{-\varsigma_i\Omega(t-\tau)}F\left(\Omega\int_{-\infty}^{\tau}\varepsilon(\tau')e^{-\varsigma_i\Omega(t-\tau')}d\tau'\right)d\tau\right). \quad (9)$$

Note that equations of this kind (i.e., with relaxation kernels) were earlier introduced phenomenologically to describe the imperfectly (inherently) elastic materials [4, 15, 16].

Equation of state (9) can be used to analyze the frequency behavior of the elastic nonlinearity of the microinhomogeneous medium. As can be seen from this equation, the dynamic action manifests itself in the nonlinear response of the medium in two ways. Firstly, this is the effect of the linear relaxation of the medium, because the linear response is the argument of the nonlinear correction $F(\varepsilon_i^{(1)})$. Secondly, the relaxation affects the response of the medium to the nonlinearity-induced secondary sources (the nonlinear correction F), which govern the nonlinearity-induced strain (or stress). These mechanisms (or, rather, components of a single nonlinear relaxation process) are essentially different. The first mechanism is universal for any nonlinear correction and is independent of the nature of the nonlinear process. The second mechanism strongly depends on the time scale of the nonlinear strain; therefore, the particular type of the nonlinearity and of the nonlinear process is significant (for example, it is important whether the process upconverts or downconverts the frequency). Nevertheless, Eq. (9) allows us to make some sufficiently general conclusions. The following estimate is valid for the relaxation integral of the function $f(t)$:

$$|f(t)| \geq \Omega\left|\int_{-\infty}^{t} f(\tau)e^{-\Omega(t-\tau)}d\tau\right|. \quad (10)$$

Then, the elastic nonlinearity of the medium containing relaxing nonlinear inclusions diminishes with increasing frequency of the action, because the argument of the nonlinear function in Eq. (9) decreases.

Below, we consider the basic features of the frequency dependence of the elastic nonlinearity for a medium with a quadratic nonlinearity, $F(\varepsilon_i) = \Gamma\varepsilon_i^2$, by examples of the generation (or demodulation) of the second harmonic and the difference frequency under a harmonic and biharmonic action on the medium.

We begin with analyzing a medium containing identical inclusions ($\varsigma_i = \varsigma$). Consider the downconversion process when the difference-frequency stress is produced under the biharmonic strain of the medium:

$$\varepsilon(t) = \varepsilon_0\cos\omega_1 t + \varepsilon_0\cos\omega_2 t. \quad (11)$$

Substituting Eq. (11) into Eq. (9) and separating the components at $\omega_d = |\omega_1 - \omega_2|$, we derive the expression for the difference-frequency stress σ_d at the difference frequency in the form $\sigma_d(\omega_d) = A_d\cos\omega_d t + B_d\sin\omega_d t = |\sigma_d|\cos(\omega_d t + \varphi_d)$, where the amplitude σ_d and the phase φ_d have the form:

$$|\sigma_d| = (A_d^2 + B_d^2)^{1/2}$$
$$= \frac{\nu\Gamma\varepsilon_0^2 E}{\varsigma^2[(1+(\omega_1/\varsigma\Omega)^2)(1+(\omega_2/\varsigma\Omega)^2)(1+(\omega_d/\varsigma\Omega)^2)]^{1/2}}, \quad (12)$$

$$\varphi_d = \arctan(\beta_d/A_d)$$
$$= \arctan\left(\frac{(\omega_d/\varsigma\Omega)[2+\omega_1\omega_2/(\varsigma\Omega)^2]}{1+\omega_1\omega_2/(\varsigma\Omega)^2-(\omega_d/\varsigma\Omega)^2}\right). \quad (13)$$

As can be seen from Eq. (12), in the static limit ω_d, $\omega_{1,2} \ll \varsigma\Omega$, the amplitude is $|\sigma_d| = \sigma_{\text{stat}} = \nu\Gamma\varepsilon_0^2 E/\varsigma^2$. When the frequencies ω_d and $\omega_{1,2}$ reach the order of the characteristic relaxation frequency $\varsigma\Omega$ of the inclusions or higher, the nonlinear response of the medium decreases: $\sigma_d \sim (\omega_1\omega_2\omega_d)^{-1}$. For a low difference frequency ($\omega_d \ll \varsigma\Omega$) and for $\omega_1 \approx \omega^2 = \omega$, Eq. (12) is simplified:

$$|\sigma_d| = \nu\Gamma\varepsilon_0^2 E[\varsigma^2[1+(\omega/\varsigma\Omega)^2]]^{-1}. \quad (14)$$

This expression shows that, when $\omega \gg \varsigma\Omega$, the amplitude behaves as $|\sigma_d| \sim \omega^{-2}$.

As can be seen from Eq. (13), the relaxation of the inclusions leads to a monotonic variation of φ_d from zero (in the quasi-static limit when ω_1, ω_2, $\omega_d \ll \varsigma\Omega$) to $\pi/2$ (when ω_1, ω_2, $\omega_d \gg \varsigma\Omega$).

To describe the frequency dependence of the elastic nonlinearity of the microinhomogeneous medium in the case of the difference frequency generation, we introduce the normalized nonlinear parameter N_d defined as the ratio of the amplitude $|\sigma_d|$ given by Eq. (14) to the amplitude value $\sigma_{\text{stat}} = \nu\Gamma\varepsilon_0^2 E/\varsigma^2$ in the static limit:

$$N_d = [1+(\omega/\varsigma\Omega)^2]^{-1}. \quad (15)$$

The parameter N_d versus frequency ω is shown in Fig. 2 (curve *1*).

Consider the process of the second harmonic generation under the harmonic action on the medium: $\sigma(t) = \sigma_0\cos\omega t$. In this case, Eq. (9) yields the expression for the stress at the double frequency, $\sigma_2 = A_2\cos 2\omega t +$

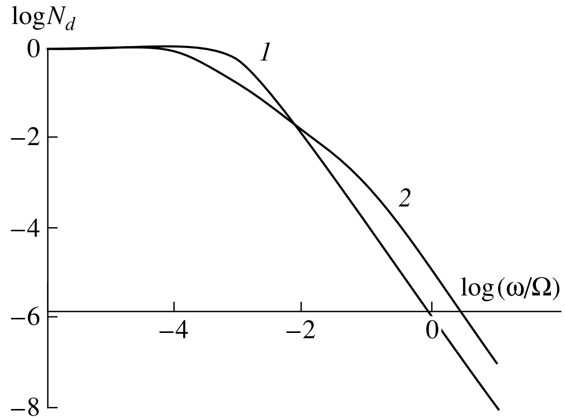

Fig. 2. Normalized nonlinear parameter N_d versus frequency ω_1 for the process of the difference frequency generation ($\omega_d/\Omega = 10^{-5}$): (*1*) a medium with identical inclusions ($\varsigma = 10^{-3}$) and (*2*) a medium with inclusions distributed in elasticity ($a = 10^{-4}$, $b = 10^{-1}$).

$B_2\sin 2\omega t$, the amplitude and phase of the stress σ_2 being determined as

$$|\sigma_2| = \nu\Gamma\varepsilon_0^2 E\frac{1}{2\varsigma^2[1+(\omega/\varsigma\Omega)^2][1+(2\omega/\varsigma\Omega)^2]^{1/2}}, \quad (16)$$

$$\varphi_2 = \arctan\left(\frac{2(\omega/\varsigma\Omega)[(\omega/\varsigma\Omega)^2-2]}{5(\omega/\varsigma\Omega)^2-1}\right). \quad (17)$$

Figure 3a shows the frequency dependences of the normalized nonlinear parameter N_2 introduced according to Eq. (15) and of the phase σ_2. Curve *1* demonstrates a rapid decrease in the parameter N_2 when the frequency ω is higher than the inclusion relaxation frequency $\varsigma\Omega$. In addition, Eq. (16) shows the above-mentioned effect of the inclusion relaxation on the nonlinearity of the medium at the frequency of the primary excitation ω and at the frequency of its second harmonic, 2ω. Note that the following inequality is valid: $N_2 \le N_d$.

In contrast to the smooth variation of the phase from 0 to $\pi/2$ in the case of the difference frequency generation, from Eq. (17), it follows that the phase φ_2 of the second harmonic changes rapidly by π in the vicinity of the frequency $\omega = \varsigma\Omega/\sqrt{5}$. This property can be used to change the frequency dependence of the parameter N_2: when the medium contains inclusions with different relaxation frequencies, their nonlinear responses superimpose, which may cause a nonmonotonic frequency dependence of the parameter N_2. Figure 3b shows the dependences of N_2 and φ_2 on frequency for a medium with inclusions of two types (their relaxation frequencies differ by an order of magnitude). These dependences demonstrate the nonmonotonic behavior mentioned above.

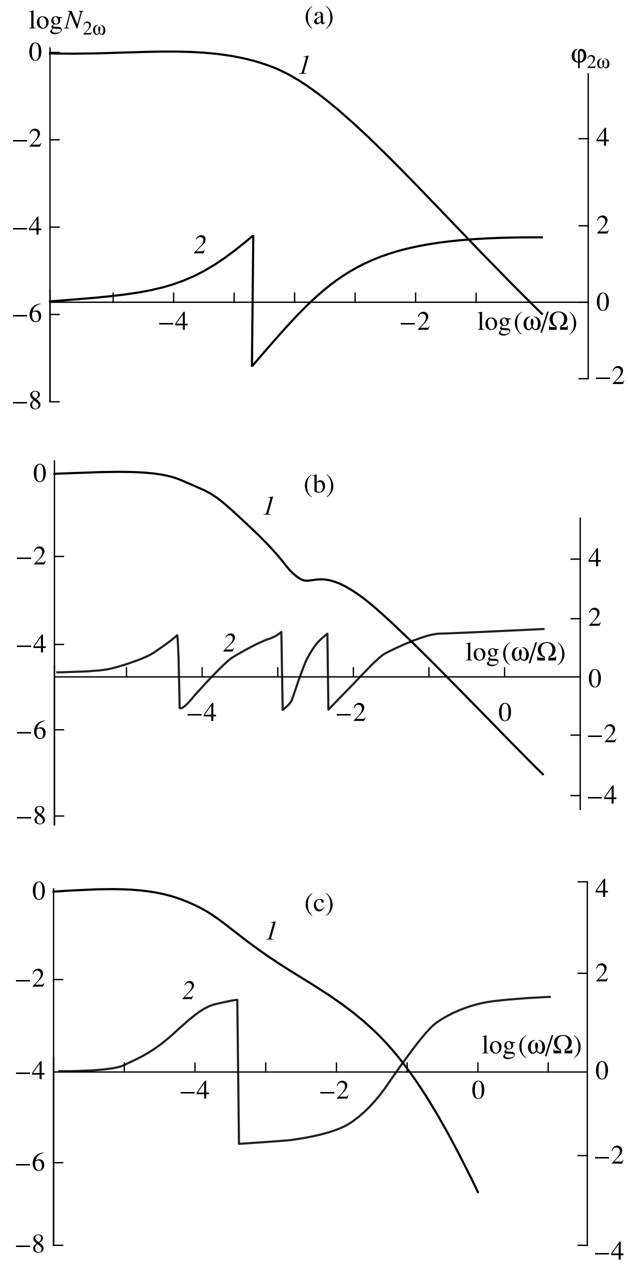

Fig. 3. (*1*) Normalized nonlinear parameter $N_{2\omega}$ and (*2*) phase $\varphi_{2\omega}$ versus frequency ω for the process of the second harmonic generation (a) in a medium with identical inclusions ($\varsigma = 10^{-3}$), (b) in a medium with inclusions of two types ($\varsigma_1/\varsigma_2 = 10^{-1}$), and (c) in a medium with inclusions distributed in elasticity ($a = 10^{-4}$, $b = 10^{-1}$).

In real microinhomogeneous media, inclusions are not identical and are characterized by a certain distribution in elasticity, $\nu = \nu(\varsigma)$, so that $\nu(\varsigma)d\varsigma$ represents the concentration of inclusions with the parameter ς within the interval $[\varsigma, \varsigma + d\varsigma]$. In general, real inclusions are also distributed in the viscosity (or in the relaxation frequency Ω), so that the inclusion distribution function must depend on the parameters ς and Ω: $\nu = \nu(\varsigma, \Omega)$. In this case, Eqs. (1)–(7) provide an evident generalization of Eq. (9):

$$\sigma(\varepsilon) = E\Bigg\{\varepsilon + \int d\varsigma \int d\Omega \nu(\Omega, \varsigma)\Omega \int_{-\infty}^{t} \varepsilon(\tau) e^{-\varsigma\Omega(t-\tau)} d\tau$$
$$- \int d\varsigma \int d\Omega \nu(\Omega, \varsigma)\Omega\varsigma \int_{-\infty}^{t} e^{-\varsigma\Omega(t-\tau)} \qquad (18)$$
$$\times F\left[\Omega \int_{-\infty}^{\tau} \varepsilon(\tau') e^{-\varsigma\Omega(t-\tau')} d\tau'\right]\Bigg\}.$$

It has been shown [9–11] that, to explain the frequency-independent behavior of the Q-factor of microinhomogeneous media, one should assume that the distribution $\nu(\varsigma)$ is sufficiently wide, the wide distribution of inclusions in elasticity being of most importance particularly for the linear dissipation–dispersion properties, while their distribution in viscosity affects the results insignificantly [11]. Since the linear and nonlinear parts of Eq. (18) contain relaxation integrals of similar structure, we will first analyze the nonlinear elasticity as a function of frequency by analogy with [9–11] allowing for the distribution of the inclusions in the parameter ς under the assumption that

$$\nu(\varsigma) = \nu_0 \ \text{for} \ \varsigma \in [a, b],$$
$$\nu(\varsigma) = 0 \ \text{for} \ \varsigma \notin [a, b]; \quad a \le \varsigma \le b < 1. \qquad (19)$$

As above, we consider the processes of the difference frequency and second harmonic generation in a medium with a quadratic elastic nonlinearity.

In the case of the difference frequency generation, one can obtain expressions similar to Eqs. (12) and (13). However, they are rather lengthy. Therefore, below, we present the formulas for the quadrature coefficients A_d and B_d derived for a low difference frequency when the primary-excitation frequencies are approximately equal ($\omega_1 \approx \omega_2 = \omega$) and $\omega_d \ll \omega$:

$$A_d = \nu_0 \Gamma \varepsilon_0^2 E \left(\frac{\Omega}{\omega}\right)^2$$
$$\times \left[\frac{\omega_d}{\Omega}\arctan\left(\frac{\varsigma\Omega}{\omega_d}\right) - \frac{\omega}{\Omega}\arctan\left(\frac{\varsigma\Omega}{\omega}\right)\right]\Bigg|_{\varsigma=a}^{\varsigma=b}, \qquad (20)$$

$$B_d = \nu_0 \Gamma \varepsilon_0^2 E \frac{\omega_d \Omega}{\omega^2} \ln\left[\frac{1 + (\omega/\varsigma\Omega)^2}{1 + (\omega_d/\varsigma\Omega)^2}\right]\Bigg|_{\varsigma=a}^{\varsigma=b}. \qquad (21)$$

These expressions also show the effect of the relaxation at both the excitation and difference frequencies; the medium is characterized by two relaxation frequencies $a\Omega$ and $b\Omega$, which correspond to the lower and higher boundaries of the inclusion distribution in elasticity, respectively. The dependence of the parameter N_d on the frequency ω is shown in Fig. 2 (curve *2*). This plot shows that $N_d = \text{const}$ for $\omega < a\Omega$. When $a\Omega < \omega < b\Omega$, we have $N_d \sim \omega^{-1}$, i.e., N_d decreases more slowly than predicted by Eqs. (12) and (14), which are derived for the medium with identical inclusions. For $\omega > b\Omega$, we have $N_d \sim \omega^{-2}$ (as for the medium with identical inclusions for $\omega > \varsigma\Omega$).

For the second harmonic, one can obtain the following expressions for the coefficients A_2 and B_2 by analogy with Eqs. (15) and (16):

$$A_2 = \nu\Gamma\varepsilon_0^2 E\left[\frac{2}{\omega/\Omega}(\arctan(\varsigma\Omega/2\omega)\right.$$
$$\left.- \arctan(\varsigma\Omega/\omega)) + \frac{1}{\varsigma[1 + (\omega/\varsigma\Omega)^2]}\right]\Bigg|_{\varsigma=a}^{\varsigma=b}, \qquad (22)$$

$$B_2 = \nu\Gamma\varepsilon_0^2 E\left\{\frac{1}{\omega/\Omega}\ln\left[\frac{1 + (\omega/\varsigma\Omega)^2}{1 + (2\omega/\varsigma\Omega)^2}\right]\right.$$
$$\left. + \frac{\omega/\varsigma\Omega}{\varsigma[1 + (\omega/\varsigma\Omega)^2]}\right\}\Bigg|_{\varsigma=a}^{\varsigma=b}. \qquad (23)$$

The effect of the inclusion relaxation at the excitation frequency and at its second harmonic can also be seen here. Figure 3c shows the parameter N_2 and the phase φ_2 versus the frequency ω. One can see that, when $\omega < a\Omega$, $N_2 = \text{const}$; in the range $a\Omega < \omega < b\Omega$, $N_2 \sim \omega^{-1}$; and for $\omega > b\Omega$, $N_2 \sim \omega^{-3}$.

Now, we consider the combined effect of the inclusion distributions in elasticity and viscosity. We assume that the inclusions are uniformly distributed in the parameters ς and Ω:

$$\nu(\varsigma, \omega) = \nu_0 \ \text{for} \ \varsigma \in [a, b], \quad \Omega \in [\Omega_a, \Omega_b],$$
$$\nu(\varsigma, \omega) = 0 \ \text{for} \ \varsigma \notin [a, b], \quad \Omega \notin [\Omega_a, \Omega_b]. \qquad (24)$$

In this case, we failed to study Eq. (18) analytically and generalize Eqs. (20)–(23), though, in principle, the solutions of interest can be obtained numerically. Here, we present the approximate analytical result describing the process of demodulation ($\omega_d = 0$):

$$\sigma_d = \frac{1}{2}\nu_0 \Gamma \varepsilon_0^2 E N_d,$$

$$N_d = \left[\left(\frac{\Omega^2}{\omega} + \frac{\omega}{b^2}\right)\arctan\left(\frac{\Omega b}{\omega}\right)\right. \qquad (25)$$
$$\left. - \left(\frac{\Omega^2}{\omega} + \frac{\omega}{a^2}\right)\arctan\left(\frac{\Omega a}{\omega}\right) + \frac{\Omega(b-a)}{ab}\right]\Bigg|_{\Omega=\Omega_a}^{\Omega=\Omega_b}.$$

Figure 4 represents the parameter N_d versus frequency ω for different values of the parameter Ω_b/Ω_a. As can be seen from Fig. 4, the additional allowance made for the distribution of inclusions in the viscosity (unlike the distribution in elasticity) weakly affects the

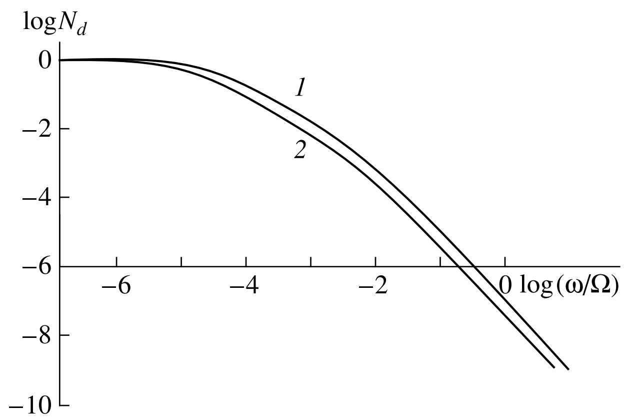

Fig. 4. Normalized nonlinear parameter N_d versus frequency ω for the process of demodulation ($\omega_d = 0$) in the medium with inclusions distributed in elasticity ($a = 10^{-4}$, $b = 10^{-1}$) and viscosity: Ω_b/Ω_a = (*1*) 10 and (*2*) 10^4.

nonlinear parameter N_d, as in the case of the results obtained in [9–11] for linear dissipation and dispersion properties of a microinhomogeneous medium.

Thus, in this paper, in the framework of a rheological model, a nonlinear dynamic equation of state of the microinhomogeneous medium containing viscoelastic inclusions is derived and, for the case of the quadratic elastic nonlinearity, the frequency dependences of the effective nonlinear parameters are determined for the processes of the difference frequency and second harmonic generation. It is shown that the frequency dependence of the nonlinear elasticity of the medium is governed by the combined effect of (i) the linear relaxation response of the inclusions at the frequency of the primary excitation and (ii) their relaxation at the combination frequencies and harmonics. Note that, though the consequences of the equation of state are analyzed for a medium with a quadratic nonlinearity, the approach developed in this paper can also be applied to media with other types of the elastic nonlinearity: cubic, different-modulus, hysteretic, etc. [4, 17, 18].

The equation of state derived above (together with the equation of motion) can be used to study various nonlinear effects that occur in the propagation and interaction of elastic waves in microinhomogeneous media. Due to the above-mentioned specific features of these media, the character of the nonlinear processes in them essentially differs from that of the nonlinear processes in homogeneous media, which can be used as a diagnostic indicator in the remote monitoring of the medium.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (project nos. 98-05-64683 and 98-02-17686) and by the Interbranch Center for Science and Engineering (project no. 1369).

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Translated by A. Khzmalyan