Elastic Waves in Media with Nonlinear Dissipation  
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Received July 5, 1996

Abstract—The paper presents a rheological model that treats a medium with nonlinear dissipation as a unidimensional chain of masses interacting through linear and nonlinear inelastic links. In the framework of this model, we derived a nonlinear wave equation and studied such effects of propagation and interaction of elastic waves as self-action of continuous harmonic and pulse waves, generation of higher order harmonics, and the effect of a strong wave on the attenuation of weak waves of other frequencies.

INTRODUCTION

Within recent years much attention in acoustic research has been directed towards nonlinear effects in structurally inhomogeneous media. As a rule, these media are characterized by a strong acoustic nonlinearity as compared to that of homogeneous materials [1, 2]. Experimental data suggest that the nonlinearity of such media is often dissipative (inelastic) rather than reactive (elastic) [3–6]. A theoretical description of phenomena observed in such media is possible perhaps only in the framework of phenomenological equations of state, since a microscopic theory of friction for solids is yet to be developed.

In this paper, we used the phenomenological dependencies of nonlinear friction of solids in contact to derive a wave equation with dissipative nonlinearity and analyze the nonlinear effects that occur in propagation and interaction of acoustic waves.

RHEOLOGICAL MODEL OF AN ELASTIC MEDIUM WITH NONLINEAR DISSIPATION

We consider a unidimensional chain of masses \( M \) and elastic elements augmented in parallel with dissipative elements, as shown in Fig. 1. Individual links of such a chain can be put in correspondence with certain structural units of the medium, e.g., granules, crystalites in polycrystals, rough surfaces of cracks, or grains in contact.

We write the equations of motion for a chain element \( n \) whose displacement from the equilibrium position will be denoted by \( U(n) \). In describing the forces with which adjacent elements act on element \( n \), we confine ourselves to the linear approximation for the elastic force \( F_1(n) \) and take into account only the nonlinearity of the friction force \( F_2(n) \):

\[
Md^2U(n)/dt^2 = F_1(n) + F_2(n), \tag{1}
\]

where \(|U(n)| \ll a \) and \( a \) is the size of a link in the chain.

For the elastic force, we have [7]

\[
F_1(n) = -\kappa[U(n) - U(n - 1)] + \kappa[U(n + 1) - U(n)], \tag{2}
\]

where the coefficient \( \kappa \) characterizes the elasticity of a link.

We assume that the dissipative force \( F_{\text{dis}}(n) \), acting between two adjacent links in the chain, is a function (in the general case nonlinear) of a relative velocity of these elements \( \Delta U = U(n + 1) - U(n) \). For macroscopic elements of the medium, we will use the known approximations of \( F_{\text{dis}}(\Delta U) \) [31] determined experimentally. Then, the linear component of the friction force \( F_2^\text{lin} = F_{\text{dis}}^\text{lin}(\Delta U) - F_{\text{dis}}^\text{lin}(\Delta U, n-1) \) acting on element \( n \) from two adjacent elements will be

\[
F_2^\text{lin}(n) = g\Delta U_{n+1,n} - g\Delta U_{n,n+1}, \quad g = \text{const}. \tag{3}
\]

The nonlinear component of friction force \( F_{\text{dis}}^\text{nl}(\Delta U) \), also acting against the relative velocity \( \Delta U(n) \), will be written in the form

\[
F_{\text{dis}}^\text{nl} = -\Phi(\Delta U) \text{sgn}(\Delta U), \tag{4}
\]

where \( \Phi(\Delta U) \) is an even, positive definite function.

Substituting the continuous coordinate \( x = na \) for a discrete number \( n \) of chain element in equations (1)–(4), we obtain in the long-wave approximation

\[
U_{xx} - c^2 U_{xx} = \alpha U_{xx} + [\Phi(U_{xx}) \text{sgn}(U_{xx})]_x, \tag{5}
\]

![Fig. 1. Rheological mode of a chain element: (1) mass, (2) elastic element, and (3) dissipative element.](image_url)
where \( c^2 = \frac{\kappa a^2}{M} \) and \( \alpha = \frac{g a^2}{M} \). (Here, for simplicity, we assume the normalization by mass \( M \) in the function \( \Phi(U) \).)

A similar equation was obtained by Nazarov [9] who modeled the stress \( \sigma \) in the medium as a sum of the elastic stress \( \sigma_1 = E \epsilon \) (\( E \) is the elasticity modulus, \( \epsilon = U \) is the deformation) and inelastic (viscous) stress \( \sigma_2 \) which in addition to a linear component dependent on the deformation rate \( \dot{\epsilon} = \frac{\partial U}{\partial \epsilon} \) contains a nonlinear component also dependent on \( \dot{\epsilon} \). In this paper, the nonlinear inelastic stress is represented by a rather common, exponential approximation [8]

\[
\Phi(U) = \gamma|U|^m, \tag{6}
\]

where \( \gamma \) and \( m \) are constant coefficients, and the dimensionality of \( \gamma \) depends on the exponent \( m \): \( [\gamma] = cm^2s^{-m} \).

For \( m > 1 \), the effective viscosity of the medium increases with the deformation rate, and, for \( m < 1 \), it decreases accordingly. The case of \( m = 0 \) corresponds to dry (Coulomb) friction [8]. The propagation of elastic waves in the medium with \( m = 0 \) has been analyzed also by Pal' mov [10] and Nikolaevskii [11].

It is worth noting that for \( m < 0 \), equation (6) is, generally speaking, inapplicable for the description of nonlinear viscous stress near \( \dot{\epsilon} = \frac{\partial U}{\partial \epsilon} = 0 \), since \( \Phi(U) = 0 \) --- \( \infty \). Therefore, for \( m < 0 \), beginning with some deformation rate \( |\dot{\epsilon}| \leq \dot{\epsilon}_0 \), the nonlinear stress must be restricted, e.g., to a small section of a dry friction:

\[
\Phi(\epsilon) = \begin{cases} 
\Phi_0 & \text{for } |\epsilon| < \dot{\epsilon}_0 \\
\gamma|\epsilon|^m & \text{for } |\epsilon| \geq \dot{\epsilon}_0,
\end{cases} \tag{7}
\]

where the constant \( \Phi_0 \) is defined from the condition of continuity of the function \( \Phi(\epsilon) \) at points \( \epsilon = \pm \dot{\epsilon}_0 \), namely, \( \Phi_0 = \gamma \dot{\epsilon}_0^m \).

For a harmonic dependence \( \epsilon = \epsilon_0 \cos \theta, \dot{\epsilon} = \Omega \), it is convenient to introduce a threshold phase (or cutoff angle) \( \theta_0 \), defined from the expression \( \dot{\epsilon}_0 = \Omega \epsilon_0 \sin \theta_0 \) or \( \sin \theta_0 = \epsilon_0 / \dot{\epsilon}_0 \), where \( \dot{\epsilon}_0 = \Omega \epsilon_0 \).

SELF-ACTION OF A HARMONIC WAVE AND GENERATION OF ITS HIGHER ORDER HARMONICS IN A NONLINEARLY DISSIPATIVE MEDIUM

We will demonstrate below that the effects of self-action of a harmonic wave and generation of its higher harmonics in dissipative-nonlinear media significantly differ from the usually considered case of reactive square nonlinearity [7].

The boundary condition for (5) will be posed in the form

\[
U(x = 0, t) = U_0 \sin \Omega t. \tag{8}
\]

It is convenient to pass from (5) to the equation

\[
U_t = \frac{\alpha}{2c^2} U_{ttt} + \frac{1}{2c^2} (\Phi(U/c) \text{sgn}(U/c)) \tag{9}
\]

where we introduced the "traveling" and "slow" variables \( \tau = (t - x/c) \) and \( x = x. \) A solution to equation (9) subject to the boundary condition (8) will be sought in the form

\[
U = U^{(\text{f})} + \Delta U, \quad \epsilon = \epsilon^{(\text{f})} + \Delta \epsilon, \tag{10}
\]

\[
U^{(\text{f})} = U^{(\text{f})}(x) \sin \theta, \quad \epsilon^{(\text{f})} = \epsilon^{(\text{f})}(x) \sin \theta = \epsilon^{(\text{f})}(x) \sin \theta, \tag{11}
\]

\[
\Delta U = \sum_{n=3}^{\infty} U^{(n)}(x) \sin (n \theta), \tag{12}
\]

where \( U^{(\text{f})} \) is the field of the fundamental frequency, \( \Delta U \) is a small correction associated with the field of higher-order odd harmonics (\( \Delta \epsilon \ll |\epsilon^{(\text{f})}| \), \( n = 3, 5, \ldots \) are the odd harmonics involved, and \( \theta = \Omega \tau \).

This form of solution (11), (12), which disregards nonlinear distortions of phase, is justified by the fact that dissipation does not produce a dependence of the wave propagation velocity on its amplitude. Substituting (10)–(12) in (9) and collecting the respective Fourier harmonics in the nonlinear term \( F(U) = \Phi(U/c) \text{sgn}(U/c) \), we obtain the following system of coupled equations for the amplitude \( U^{(i)}(i = 1, 3, 5, \ldots) \):

\[
U_t^{(i)} = -\beta U^{(i)} + \frac{1}{2} c^2 A^{(i)}, \tag{13}
\]

where

\[
\beta = \frac{\alpha \Omega^2}{2c^3}
\]

and

\[
A^{(i)}(x) = \pi^{-1} \int F[U^{(i)}(x)] \sin (n \theta) d\theta. \tag{14}
\]

In order to determine the coefficients \( A^{(i)}(x) \), we expand \( F \) in the integrand of (14) into a Taylor series

\[
F(U) = F(U^{(1)}_{tt}) + F^{(1)}(U^{(1)}_{tt}) \Delta U_{tt} + \ldots. \tag{15}
\]

Representing the functions \( F(U^{(1)}_{tt}) \) and \( F^{(1)}(U^{(1)}_{tt}) \) in this expression as Fourier components, we obtain from (13) the following system of equations for the amplitude of the fundamental and its harmonics:

\[
U_t^{(1)} = -\beta U^{(1)} - c^{-2} \frac{1}{2} c^2 B^{(1)}(U^{(1)}), \tag{16}
\]
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\[ U_x^{(e)} = -(\Omega^2 n^2/2c^3)(\alpha + C^{(0)}(U^{(1)}))U_x^{(e)} - (1/2c^2)B^{(s)}(U^{(1)}), \]

where

\[ B^{(s)}(U^{(1)}(x)) = \pi^{-1} \int_0^\pi F[\Omega \varepsilon_1(x) \sin \theta] \sin \theta d\theta, \]

\[ B^{(s)}(U^{(1)}(x)) = \pi^{-1} \int_0^\pi F[\Omega \varepsilon_1(x) \sin \theta] \sin (n\theta) d\theta, \]

\[ C^{(0)}[U^{(1)}(x)] = (2\pi)^{-1} \int_0^\pi F[\Omega \varepsilon_1(x) \sin \theta] d\theta. \]

We touch on the effect of self-action only briefly, because this topic has been discussed in detail by Nazarov [9], Pal'mov [10], and Nikolaevskii [11]. Using the approximation (7) and notations introduced in (10) and (12), we rewrite (18) as

\[ B^{(1)} = \begin{cases} 
(4/\pi)\gamma(\Omega \varepsilon_b)^n & \text{for } \dot{\varepsilon}_1 \leq \dot{\varepsilon}_b \\
(4/\pi)\gamma(\Omega \varepsilon_b)^m \left(1 - \cos \theta_{\text{th}}\right) & \\
+ \left(\varepsilon_1 \varepsilon_b / \theta_{\text{th}}\right)^{n/2} (\sin \theta)^{m+1} d\theta & \text{for } \dot{\varepsilon}_1 > \dot{\varepsilon}_b.
\end{cases} \]

(21)

We note that, for \( m > -2 \) and \( \varepsilon_b \) (or cutoff angle \( \theta_{\text{th}} \)) tending to zero, this expression does not acquire any singularity; therefore, at \( \dot{\varepsilon}_b = 0 \), equations (21) and (16) can be reduced to a simpler form obtained by Nazarov [9] for \( m > -2 \) from a dissipative force approximation in the form of (6) without restrictions at zero.

For \( m \leq -2 \), equation (21) can be readily analyzed for \( \varepsilon_1 \leq \varepsilon_b \) and for \( \varepsilon_1 > \varepsilon_b \). In the last case, at \( m = -2 \), equation (16) can be approximately written as

\[ U_x^{(1)} = -\beta U^{(1)} - \delta \ln[2\Omega U^{(1)}/\pi \varepsilon_\text{th}](U^{(1)})^{-2}, \]

\[ \delta = 2\gamma/\pi \Omega^2 c^2, \]

and, for \( m < -2 \), in the form

\[ U_x^{(1)} = -\beta U^{(1)} + \delta (U^{(1)})^{-2}, \]

\[ \delta = \gamma m \Omega^{-2} [\varepsilon_\text{th}]^{m+2}/(m + 2)c^2. \]

(23)

It is worthwhile to note that, for \( m < -2 \), the amplitude dependence of the nonlinear summand saturates, so that its decay law becomes universal \((-U^{(1)})^{-2}\). The solution of equations of the type (23) has the form (12), which is valid for any \( m > 1 \), waves with larger amplitudes decay more strongly. (Such an effect was observed in experiments with marble and granite [4, 5].) For \( m < 1 \), the attenuation of a wave abates as its amplitude increases. Such an effect (bleaching of the medium) was observed in experiments with dry and wet river sand [4, 6].

Now we turn to generation of higher harmonics. From equation (17) it follows that the field of the initial wave excites the \( n \)th odd harmonic, owing to the presence of the component in the spectrum of the exciting force (coefficient \( B^{(0)}[U^{(1)}(x)] \)), and also changes the coefficient of absorption of the weak wave (coefficient \( C^{(0)}[U^{(1)}(x)] \) in equation (17)). As a result, from equation (17) we obtain for the amplitude of harmonic \( n \)

\[ U^{(n)}(x) = 1/2c^2 \left[ \int_0^\infty \exp \left( \frac{\xi}{2} \right) \right] \]

\[ \times \exp \left[ -\frac{\xi}{\omega} \right] \Gamma^{(n)}(x), \]

(25)

where the absorption coefficient \( \Gamma^{(n)} \) of this harmonic has the form

\[ \Gamma^{(n)}(x) = (\Omega^2 n^2/2c^3)(\alpha + C^{(0)}[U^{(1)}(x)]). \]

(26)

The asymptotic representation of equation (25) is universal at the initial step of generation, when one may neglect the absorption of the fundamental and its harmonics in the medium. In this approximation, the magnitude of harmonic \( n \) will increase linearly with distance as \( U^{(n)} = nB^{(0)}[U^{(1)}(x = 0)] \). In the case of exponential dependence of nonlinear dissipation, for a positive integer, and odd \( m \), the spectrum of driving force will contain only odd harmonics with numbers \( 3 \leq n \leq m \); whereas, with other positive \( m \), it will contain odd harmonics with \( n \geq 3 \) and \( U^{(n)} = [U^{(1)}(x = 0)]^m \). For negative \( m \), it follows from equation (19) that the driving force decreases as the magnitude of the fundamental increases. This behavior considerably differs the considered case from traditional problems of nonlinear acoustics in which elastic nonlinearity manifests more strongly as the driving force increases. For \( -2 < m < 0 \), integral (19) of the driving force converges and has the form

\[ B^{(s)} = 2\pi^{-1} \gamma \varepsilon_1 \int_0^\pi \sin \theta \sin (n\theta) d\theta. \]

(27)

This expression suggests that when \( \dot{\varepsilon}_1 \) tends to zero, the driving force becomes infinite at higher order harmonics. In order to cope with this divergence for \( m < -2 \),
we introduced above the dissipative force approximation (7). At small cutoff angles \( \Theta_n \ll 1 \), for \( m > -2 \), this approximation yields the same result as (27) obtained from equation (6), and, for \( m \leq -2 \), it is not hard to obtain the following approximations:

\[
B^{(m)} = \begin{cases} 
\frac{4}{\pi} \gamma n \Omega^2 \varepsilon_1^2 \left[ \frac{1}{2} \ln \frac{\varepsilon_{th}}{\varepsilon_1} \right] & m = -2 \\
\frac{4}{\pi} \gamma n \left[ \frac{1}{2} - \frac{1}{m + 2} \right] \Omega^m \varepsilon_{m} \left( \frac{\varepsilon_{th}}{\varepsilon_1} \right)^2 & m < -2.
\end{cases}
\]  

(28)

Thus, at large amplitudes (\( \varepsilon_1 \gg \varepsilon_{th} \)), equation (28) also demonstrates how the driving nonlinear force decreases as \( \varepsilon_1 \) increases, although for \( m < -2 \), the degree of such reduction is saturated (\( B^{(m)} \sim \varepsilon_1^{-2} \)).

At small amplitudes (\( \varepsilon_1 \leq \varepsilon_{th} \)) from (7) and (19) we have

\[
B^{(m)} = -4 \Phi_0/\pi n = -4 \gamma \Omega^m \varepsilon_{m}^2/\pi,
\]  

(29)

so that the driving force does not depend on \( \varepsilon \) and remains finite at any amplitude of the initial wave however small. This conclusion implies that the approximation (7) is too rough for the evaluation of the nonlinear force at small (subthreshold) values of \( \varepsilon_1 < \varepsilon_{th} \). It can be refined, e.g., by letting

\[
\Phi(\varepsilon) = \begin{cases} 
(\varepsilon/\varepsilon_{th})^s \Phi_0 & \text{for } \varepsilon \leq \varepsilon_{th} \\
\gamma \varepsilon^m & \text{for } \varepsilon > \varepsilon_{th},
\end{cases}
\]  

(30)

where \( s > 0 \). As \( s \rightarrow 0 \), this expression passes into (7).

In this case, at small \( \varepsilon_1 \leq \varepsilon_{th} \), from (19) it follows

\[
B^{(n)} = \gamma \Omega^m \varepsilon_{m}^2 \left( \frac{\varepsilon_{th}}{\varepsilon_1} \right)^s, \\
A^{(n)} = \frac{(2/\pi) \sin(\pi n/2) \Gamma(s + 1)}{\Gamma(s + n/2 + 1) \Gamma(s - n/2 + 1)}.
\]  

(31)

Thus, when the function \( \Phi(\varepsilon) \) has a falling segment, at small amplitudes (\( \varepsilon_1 < \varepsilon_{th} \)), the amplitude of higher order harmonics \( U^{(m)} \) sharply increases to reach a maximum at \( \varepsilon_1 \sim \varepsilon_{th} \) and sets about a gradual descent for \( \varepsilon_1 > \varepsilon_{th} \).

NONLINEAR ATTENUATION OF A WEAK WAVE IN THE FIELD OF A STRONG PUMP WAVE

Now, we consider the effect of strong harmonic pump wave \( U_0(x, t) = U_0(x) \sin(\Omega t + Kx) \) with frequency \( \Omega \) on the attenuation of a weak wave \( U_2(x, t) = U_2(x) \sin(\omega t - kx) \) with an incomparable frequency \( \omega \) (\( \omega/\Omega \neq p/q \), where \( p \) and \( q \) are integers). In this case, for the amplitude \( U_2 \) of the weak wave, equation (5) gives

\[
U_{2, r} = -(\beta + \beta_{m}(\varepsilon_1)) U_2,
\]  

(32)

where

\[
\beta = \omega \omega' / 2c^2, \\
\beta_{m}(\varepsilon_1) = (\omega/2c^2) C^{(0)}(\varepsilon_1), \\
\pi/2 \\
C^{(0)}(\varepsilon_1) = (2/\pi) \int F(\Omega \varepsilon_1(x) \sin \theta) d\theta.
\]  

(33)

The appearance of the term \( \beta_{m}(\varepsilon_1) U_2 \) in equation (32) implies that the strong wave \( U_0 \) changes the dissipation of the weak wave \( U_2 \), independently of the direction of propagation of these waves (curvilinear or opposite). We consider the cases of different signs of exponent \( m \) in the exponential approximation of the dissipative function. In agreement with expressions (33) and (6) for the nonlinear dissipation coefficient, for \( m \geq 0 \), we obtain

\[
\beta_{m}(\varepsilon_1) = \frac{m \omega^2 \Gamma(m/2)}{4\pi^{1/2} c^5 \Gamma(m/2 + 1/2)} \varepsilon_{m}^{m-1}.
\]  

(34)

In particular,

\[
\beta_{m}(\varepsilon_1) = \begin{cases} 
3 \omega^2 \varepsilon_1^2 / 8c^3 & \text{at } m = 3 \\
\pi^{-1/2} \omega^2 \varepsilon_1^3 / c^3 & \text{at } m = 2.
\end{cases}
\]  

(35)

Equation (34) suggests that, for positive \( m \), the coefficient \( \beta_{m} \) is also positive; that is, the pump wave increases the attenuation of the weak wave. This effect was observed in some metals and rocks [3, 5].

In the case of a downshifting segment in the dependence \( F_{\beta m} \) on \( \varepsilon \), the coefficient \( \beta_{m} \) can be negative and thus lead to a dissipative instability of the weak wave in the field of the strong wave. Indeed, for \( \beta_{m} \) with \( m \leq 0 \) and an exponential approximation (7) of the dissipative force with a restriction at zero, we find that, in the case when \( \varepsilon_1 \) exceeds the threshold value \( \varepsilon_{th} \), the expression for \( C^{(0)} \) can be reduced to the form

\[
C^{(0)} = 2\pi^{-1} \gamma \varepsilon_{1}^{-m} \left[ \frac{\pi/2}{\varepsilon_{th}} + m \int_{\varepsilon_{th}}^{\varepsilon_{1}} \sin^{-m-1} \theta d\theta \right].
\]  

(36)

In particular, for some integer \( m \), we obtain

\[
C^{(0)} = \begin{cases} 
2\pi^{-1} \gamma \varepsilon_{1}^{-1}, & m = 0 \\
2\pi^{-1} \gamma \varepsilon_{1}^{-1} \tan(\theta_{th}/2), & m = -1 \\
2\pi^{-1} \gamma \varepsilon_{1}^{-1} [1/2 + \ln|\tan(\theta_{th}/2)|], & m = -2.
\end{cases}
\]  

(37)

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For $m < -2$, with allowance for the fact that the main contribution to integral (36) is due to the neighborhood of $\theta_m$, we obtain an approximate expression for $C^{(0)}$:

$$C^{(0)} = -2\pi^2 \gamma m (\varepsilon_m^2 / \varepsilon_0)^{(m-1)/(m+2)}.$$  \hfill (38)

These expressions suggest that at a rather steep slope of the decaying curve (when $m \leq -2$), the coefficient of nonlinear dissipation can become negative.

Thus, in agreement with formulas (37) and (39), at a downsloping segment of the curve, the magnitude of nonlinear negative absorption must reach a maximum at moderate $\varepsilon$, exceeding $\varepsilon_m$, and a further increase of $\varepsilon$ will decrease $C^{(0)}$. The qualitative profile of the dependence of $C^{(0)}$ on the amplitude $\varepsilon$ will, naturally, retain its main features with any other method of introduction of the downsloping segment into $\Phi(\varepsilon)$, as well.

**IMPULSE TOMOGRAPHY OF DISSIPATIVE NONLINEARITY PARAMETER**

In view of a high sensitivity of both elastic and inelastic nonlinear parameters to changes in the medium structure, their spatial reconstruction is of practical significance. The tomographic methods proposed for these parameters are based on the phase and amplitude modulation obtained by a probing wave in its interaction with a strong pulse [13, 14]. Below we use an example of interaction of countercurrent plane longitudinal waves (a probing continuous wave and a pump pulse) to analyze the recovery of the nonuniform pattern of medium's dissipative nonlinear properties; this reconstruction may be of interest, e.g., for seismological applications.

Let a medium with dissipative nonlinearity propagate two opposite waves: a strong low-frequency pump pulse $U_1(x, t)$ and a weak continuous high-frequency wave $U_2(x, t)$.

$$U_1(x, t) = U_1 \Pi(x - D + ct) \sin(\Omega t + K(x - D)),$$

$$U_2(x, t) = U_2(x) \sin(\omega t - kx),$$  \hfill (39)

where $\Pi(x - D + ct)$ is the rectangular envelope of a pulse of length $T \gg (2\pi/\Omega)$, $D$ is the probing path length, and $\omega \neq n\Omega$ ($n$ is an integer). Substituting $U = U_1(x, t) + U_2(x, t)$ in equation (5) and taking into account that $|\varepsilon_1| \gg |\varepsilon_2|$, we obtain, similar to (32), an equation describing the attenuation of a probing wave in the field of a strong pump pulse:

$$U_2(x) = -[\beta + \beta_m \Pi(x - D + ct)] U_2,$$  \hfill (40)

where $\beta = \varepsilon_0^2/2c^3$, $\beta_m = (\varepsilon_0^2/2c^3)C^{(0)}(\Omega \varepsilon_1)$, and the coefficient $C^{(0)}(\Omega \varepsilon_1)$ is defined by formulas (33)-(38).

For an inhomogeneous medium, the quantities $c$, $\beta$, and coefficients $\gamma$ and $m$ in the expression for $C^{(0)}(\Omega \varepsilon_1)$ depend on coordinate $x$. Assuming that the pulse length $cT$ is small compared to the characteristic size of inhomogeneity, we find the nonlinear attenuation of the probing wave produced by the strong pump pulse is

$$\chi_0(x_0) = \beta(x_0) cT, \quad x_0 = D - ct.$$  \hfill (41)

This expression defines the spatial distribution of the coefficient $m(x_0) = 1 - \exp[-\chi(x_0)]$ of amplitude modulation of the probing wave. Assuming $\chi_0(x_0) \ll 1$, we obtain

$$m(x_0) = 1 - \exp[-\chi(x_0)] = \chi_0(x_0).$$  \hfill (42)

Thus, the parameter characterizing a medium's inhomogeneity is the quantity $\beta_0 = \varepsilon_0^2 C^{(0)}(\Omega \varepsilon_1)/2c^3$, and information about medium inhomogeneity is carried by the amplitude modulation index of the probing wave. Different media are characterized by different wave propagation velocities and parameters of dissipative acoustic nonlinearity; velocities typically differ by several percentage points, or seldom by several tens of percent, whereas the nonlinear parameters of different media differ by orders of magnitude. These magnitudes promise a high contrast of reconstructed distributions.

**PROPAGATION OF A PULSE IN MEDIA WITH DISSIPATIVE NONLINEARITY**

Now, we consider the distortion of the profile of pulse trains caused by the dependence of medium's absorption on the disturbance amplitude. The boundary condition will be imposed in the form

$$U_0(x = 0, t) = A_0 \exp(-t^2/T^2) \cos(\omega_0 t), \quad \omega_0 T \gg 1.$$  \hfill (43)

In the linear approximation ($\Phi = 0$), the solution of equation (9) has the form [7]

$$U_1(x, \tau) = (4\pi \delta x)^{-1/2} \int U_0(\tau') \exp \left( \frac{(t - \tau')^2}{4\delta x} \right) d\tau',$$  \hfill (44)

where $\delta = \alpha/2c^3$.

Substituting (43) in (44) yields $U_1(x, \tau) = A(x, \tau) \cos \omega_0 \tau$.

$$A(x, \tau) = \frac{A_0}{(1 + \mu x)^1/2} \exp \left( \frac{\tau^2}{T^2} \frac{1}{1 + \mu x} \right) \times \exp \left( -\frac{\omega_0^2 \delta x}{1 + \mu x} \right).$$  \hfill (45)
where $\mu = 4\pi \delta/T^2$, and $\omega = \omega_0/(1 + \mu \chi)$.

We will seek the solution of the nonlinear equation (9) in the form

$$U(x, \tau) = U(x, \tau)F(x, \tau),$$

(46)

where the unknown slow modulation function $F(x, \tau)$ is positive and satisfies the boundary condition $F(x = 0, \tau) = 1$. Substituting (46) in (9) and taking into account that $U(x, \tau)$ is the solution in the linear approximation, we obtain an equation for $F(x, \tau)$:

$$U_l(x, \tau)F(x, \tau) = (\gamma/2c^2)(|U_lF|_0)^{m\sigma}sgn(U_lF).$$

(47)

Recognizing that the pulse envelope is a slow function ($\tau \omega_0 \gg 1$), after straightforward algebra, we obtain an equation for $F(x, \tau)$:

$$F_e = -\nu F A^{m-1},$$

(48)

where $\nu = \chi \omega_0 b/c^2$, and $b$ is the Fourier coefficient of the first harmonic on the right-hand side of equation (47). In particular, for $m > -2$, we have

$$b = -\pi^{-1}\int_{-\pi}^{\pi} \cos^{m+1} \theta d\theta = \frac{2}{\pi^{2/2} \Gamma(m/2 + 3/2)} \frac{\Gamma(m/2 + 1)}{\Gamma(m/2 + 3/2)}.$$ 

(47)

For $m \leq -2$, we need to limit the dissipative force near $\epsilon = 0$ as before (see (7)). In this case, the variation of $b$ does not matter and we shall not take it into account.

In view of $F(x = 0, \tau) = 1$, the solution of equation (48) has the form

$$F = \left[1 - \mu(1 - m) \int_0^x A^{m-1}(x, \tau)dx\right]^{1/1-m},$$

(49)

and the pulse envelope $B(x, \tau)$ is given by the expression

$$B(x, \tau) = A(x, \tau)F(x, \tau) = A(x, \tau)\left[1 - \mu(1 - m) \int_0^x A^{m-1}(x, \tau)dx\right]^{1/1-m},$$

(50)

This solution cannot be analyzed for the general case; therefore, we consider the case of $\mu x \ll 1$, when we may neglect the linear broadening of the pulse and a shift of its carrier frequency, that is, when

$$A(x, \tau) = A_0 \exp(-(\tau/T^2)) \exp(-\omega_0^2 x),$$

(51)

Substituting this expression in equation (49), we obtain

$$F^{1-m}(x, \tau) = 1 + (\nu A_0/\omega_0^2 \delta) \times \exp\left(-(\tau/T^2)(m-1)\right) \{1 - \exp(-\omega_0^2 \delta x(m-1))\}.$$ 

(52)

From this expression it is not hard to understand the behavior of the pulse envelope $B(x, \tau)$. To be more specific, in a medium with $m > 1$, the pulse will suffer nonlinear clipping and relative broadening due to a more intense dissipation near its maximum. For $m < 1$, one will observe a relative pulse sharpening due to an increased absorption (and vanishing at a finite distance) of the small-amplitude field at pulse edges. These distortions are illustrated in Fig. 2.

CONCLUSION

In order to estimate the amplitudes of waves at which the above phenomena could be observed, we refer to experimental data on the friction of solids [8]. They indicate that the characteristic threshold velocity $V_{th}$ (preceding the downslope) lies in the range $10^{-7}$ to $10^{-4}$ cm/s, which corresponds to $\epsilon_0 \sim 10^{-13}$ to $10^{-9}$. The relative velocities of structural blocks of size $a$ have relative velocities of the order of $(Ka)^{1/2}$, where $(Ka)^{1/2}$ is the wave velocity at the fundamental harmonic. For example, at $(Ka) \sim 10^{-2}$, we obtain an estimate for wave amplitudes $\epsilon_0$ at which the downslope segment begins:

$$\epsilon_0 \sim (Ka)^{1/2} V_{th}/c \sim 10^{-11} \sim 10^{-7}.$$ 

This range corresponds to experimentally feasible amplitudes of acoustical waves.

ACKNOWLEDGMENTS

This work was financially supported by the Russian Foundation for Fundamental Research under grants nos. 95-02-06411 and 96-05-64439.

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ACOUSTICAL PHYSICS Vol. 44 No. 3 1998


Translated by M.G. Edelev