The Limiting Value of the Parameter of Elastic Nonlinearity in Structurally Inhomogeneous Media

I. Yu. Belyaeva and V. Yu. Zaitsev
Institute of Applied Physics, Russian Academy of Sciences, ul. Ul'yanova 46, Nizhni Novgorod, 603600 Russia
e-mail: vyuzai@iup.appl.sci-nnov.ru
Received December 23, 1997

Abstract—The limiting possibilities for the increase in the nonlinear parameters of microinhomogeneous media are analyzed on the basis of the hypothesis assigning the nonlinear properties of microinhomogeneous media (such as bubbleous liquids, porous, granular, and cracked media) to a single structural mechanism. It is established that the limit to the increase of nonlinear parameters is determined by the ratio between the linear elastic moduli of the host material and the "soft" inclusions (defects) rather than by the nonlinear properties of the inclusions. The estimated limiting value of the nonlinear parameter is practically independent additional factors (the resonance properties and spatial orientation of soft defects and the multilevel internal structure of the medium). The physical limit to the elastic nonlinearity of any microinhomogeneous medium is caused by the limitation imposed on the values of the ratio between the elastic moduli of the host material and the soft inclusions by the atomic structure of elastic bodies. Since the ratio of the elastic moduli can vary within four to five orders of magnitude, the maximal possible value, for example, of the quadratic parameter of nonlinearity is $10^2 - 10^3$, which agrees well with the experimental data.

INTRODUCTION

In the recent years, a broad spectrum of so-called structurally inhomogeneous media was revealed. These media are characterized by anomalous (compared to homogeneous materials) nonlinear acoustic properties and, from the structural point of view, by the presence of various types of inclusions such as grains, pores, and cracks [1, 2]. It was found that these inclusions (defects) are responsible for the nonlinear properties of the media. Therefore, theoretical studies of a structurally determined nonlinearity are often performed in terms of the models of strongly nonlinear inclusions that contribute to the average nonlinearity of the material. In this connection, a question arises: What is the theoretical limit to the increase in the nonlinearity caused by the structural inhomogeneity of the medium? Such limit should appear as a result of the physical limitations, such as the limitation imposed on the minimum elastic wavelength relative to the interatomic distance in the atomic lattice. Until recently, the physical models of structurally induced nonlinearity were developed separately for the media with different kinds of inhomogeneities (bubbleous liquids [3] or porous [4–6], granular [7, 8], and cracked [9, 10] media). For these media with different structural characteristics, the expected values of their nonlinear parameters were estimated, but the possible physical limits to their nonlinear properties were never discussed. In this paper, we analyze the limits to the increase of nonlinearity in terms of a single basic structural mechanism [11, 12] responsible for the increase of the nonlinear parameters of the aforementioned and similar types of media.

STRUCTURAL MECHANISM AND BASIC PROPERTIES OF INCLUSIONS AFFECTING THE NONLINEAR PROPERTIES OF THE MEDIUM

A generalized physical model of the structural mechanism underlying the increase of the elastic nonlinearity in microinhomogeneous media was considered by Zaitsev [11, 12]. Using this model with the appropriately selected parameters, one can derive the expressions for the coefficients of nonlinearity of the media with specific types of defects. The results obtained by Zaitsev showed that, despite the existence of a wide variety of highly nonlinear structurally inhomogeneous media, there is only one common structural feature responsible for their nonlinear properties and typical for all these materials, namely, the presence of structural components with widely different linear elastic characteristics. The structural mechanism that accounts for the increase of nonlinearity can be described as follows: the presence of soft defects (inclusions) may give rise to high local strain and stress values, which, at sufficiently high concentrations of such defects, will result in anomalous nonlinear behavior of the microinhomogeneous specimen at small average strains. The individual nonlinear properties of the host material and the soft inclusions can be fairly weak. The analysis performed by Zaitsev [11, 12] was complemented in the subsequent publication [13] by taking into account the effect of the hierarchical (multilevel) structure of a microinhomogeneous medium on its nonlinear properties. The general conclusion derived from
the aforementioned publications is that the limiting level for the increase of the nonlinear elastic parameters of a structurally inhomogeneous medium is determined (correct to factors of the order of unity) by the maximum possible difference between the linear elastic moduli of the defect-free "rigid" host medium and the soft inclusions.

However, in the previous analysis of the medium [11, 12] as well as in the analysis of one of its structural levels in the hierarchical model [13], the presence of only two components characterized by widely different compressibilities and fixed values of elastic moduli was considered, while, in real microinhomogeneous media such as rock or polycrystalline materials, the elastic moduli of the inclusions distributed over the host material can represent a wide spectrum of values. Besides, in the previous studies, the deformation of the elastic microscopic inclusions was assumed to be quasistatic (i.e., the possibility of a natural resonance of inclusions at the external frequency was neglected), and the response of the inclusions to stress was assumed to be isotropic. Under these assumptions, the analysis could be performed by using a one-dimensional model of the medium. To apply the conclusions of the previous studies [11–13] to real media, it is necessary to analyze the effect of the aforementioned factors (the variety of the elastic characteristics, the resonance properties, and the anisotropy of the microscopic inclusions) on the limiting values of the nonlinear parameters of a structurally inhomogeneous medium. Below, we consider the roles of these factors.

EFFECT OF THE VARIETY
OF ELASTIC PROPERTIES
OF THE MICROSCOPIC INCLUSIONS

According to Zaitsev [12], we consider a chain of concentrated masses $M$ linked to each other by elastic elements with equal (for simplicity) unperturbed lengths $L \ll \Lambda$, where $\Lambda$ is the acoustic wavelength. We assume that most of the elastic elements have the coefficient of elasticity $\kappa$, and the rest have the coefficients of elasticity $\kappa_i$ such that the values of the relative compliance $q_i = \kappa_i/\kappa$ are distributed over a wide range, $q_i \in [1; \infty]$. The linear density of all links of the chain is characterized by the quantity $N$, and the concentration of inclusions of the $i$th type is $N_i$. For simplicity, we assume that the cross-section of the chain is of unit area. Then, the quantities $N$ and $N_i$ can be considered as the volume concentrations in the medium, and the elastic force acting between the elements corresponds to the stress $\sigma$. In the case of a large number of inclusions, it is convenient to introduce a function describing their distribution in the compliance $q$: 

$$ N(q) = N^0 f(q), \quad \text{where} \quad q \in [a, b], $$

$$ N^0 = \sum_{i=1}^{b} N_i \quad \text{and} \quad \int_{a}^{b} f(q) dq = 1. $$

(1)

In the long-wave (quasistatic) approximation, the elastic stresses applied to the adjacent stiff and compliant elements are evidently equal: $\sigma = \kappa X = \kappa_i X_i$, where $X$ and $X_i$ are the variations in the lengths of the host links and inclusion elements, respectively. Then, the total deformation of the chain consisting of $N$ elements will be expressed as

$$ X' = \sum_{i=1}^{N^0} X_i $$

$$ = \kappa \left[ 1 + \frac{N^0}{N} \int_{a}^{b} f(q) dq - 1 \right]. $$

Dividing these equations by the initial length of the chain $NL$, we obtain an expression relating the total relative deformation to the relative deformation $XL$ of individual stiff elements:

$$ \varepsilon = \frac{X}{L} \left[ 1 - \nu + \nu \int_{a}^{b} f(q) dq \right], $$

(2)

where $\nu = N^0/N$ is the relative number of soft inclusions.

To consider the nonlinear corrections introduced by the defects (inclusions) into the stress–strain relation describing a microinhomogeneous medium, it is necessary to refine the nonlinear characteristics of the elastic elements. We assume that the material of each elastic element is characterized by a weak deviation from the linear Hooke law $\sigma = E(1 + F(\varepsilon))$, where $F(\varepsilon) \ll 1$ and $E = \kappa/L$. As a common and illustrative example (as in [12]), we consider a nonlinear function of the power type $F(\varepsilon) = \Gamma^{\alpha} \varepsilon^{\alpha-1}$, where $\Gamma^{\alpha} \sim 1$ is a dimensionless coefficient of nonlinearity of the $\alpha$th order (for simplicity, we assume that its value is the same for the compliant and the stiff elements). Then, for the stiff elements, we can write

$$ \sigma = E(\varepsilon + \varepsilon^{n} \Gamma^{(n)}), $$

(3)

and for the inclusions, we have

$$ \sigma = E_i(\varepsilon + \varepsilon^{n} \Gamma^{(n)}), $$

(4)

where the linear elastic moduli $E$ and $E_i$ are evidently related as $E/E_i = \kappa/\kappa_i = q_i$.

To determine the nonlinear parameters of the chain with soft inclusions, we use the well-known relation

ACOUSTICAL PHYSICS Vol. 44 No. 6 1998
between the stress $\sigma$ and the elastic energy density $W$ in the medium [14]:

$$\sigma = \partial W/\partial e. \tag{5}$$

In an inhomogeneous chain, the average energy density $W$ is related to the elastic energy of the deformed elements of the host material $W^m$ and the energy of the inclusion elements $W^i$:

$$W = (N - N_i^0)W^m + N_i^0 W^i, \tag{6}$$

where the values of the elastic energy of individual elements are determined by their deformations $X$ and $X_i$:

$$W^m = \int (M/L)(\xi/L + \Gamma^m_r)ξ/L^m dξ, \tag{7}$$

$$W^i = \int (M/L)(\xi/L + \Gamma^i_r)ξ/L^i dξ. \tag{8}$$

Using expressions (1) and (4)–(8), we obtain an expression describing the nonlinear stress–strain relation in a medium containing inclusions with elastic moduli distributed over a wide range:

$$\sigma = \frac{M}{1 - ν + V \int q f(q) dq} \left(1 + ε^{n - 1} \Gamma^{m - 1} \right)^{1/2} \int \left(1 - ν + V \int q f(q) dq\right)^n \left(1 - ν + V \int q f(q) dq\right)^s. \tag{9}$$

If we introduce the coefficients $B$ and $A$ characterizing the effect of inclusions on the linear and nonlinear properties of the medium, respectively, the expressions for these coefficients will have the form:

$$B^{(n)} = \left[1 + V(t - 1)\right]^{-1}, \tag{10}$$

$$A^{(n)} = \frac{1 + V(s - 1)}{\left[1 + V(t - 1)\right]^n}. \tag{11}$$

where $s = \int q^n f(q) dq$ and $t = \int q f(q) dq$. From (11) it follows that the maximal value of the parameter of nonlinearity of the medium $\Gamma^{m - 1}$ is attained at the optimal concentration of defects:

$$\nu_{opt} = \frac{1}{n - 1} \frac{1}{t - 1} - \frac{n - 1}{n - 1} \frac{s - 1}{(n - 1)(t - 1)} = \frac{1}{(n - 1)(t - 1)}. \tag{12}$$

and the corresponding value of the coefficient $A^{(n)}(\nu_{opt})$ characterizing the increase in the nonlinearity is determined by the formula

$$A^{(n)}(\nu_{opt}) = \frac{(n - 1)^{n - 1}}{n^{n - 1}} q^{n - 1} - q^{n - 1} \gg 1, \tag{13}$$

The analysis of expression (13) shows that the coefficient characterizing the nonlinearity increase will reach its maximum, if the function $f(q)$ will be as narrow as possible and shifted toward the smaller values of the coefficients of elasticity of the soft inclusions, i.e., if $f(q)$ will have the shape of the delta-function, which corresponds to the presence of soft inclusions with only one fixed value of the elastic modulus, namely, the minimal possible value. In such a case, expression (13) passes into the formula obtained by Zaitsev [12] for the maximal value of the coefficient characterizing the increase in the nonlinearity of the medium:

$$A^{(n)}(\nu_{opt}) = \left(\frac{n - 1}{n}\right)^{n - 1} q^{n - 1} - q^{n - 1} \gg 1, \tag{14}$$

which yields $\Gamma_{medium}^{m} = A^{(n)}(\Gamma^{m}) \gg \Gamma^{m}$.

Thus, by taking into account the variety of the elastic properties of the soft inclusions (defects), we obtain the same estimate for the maximal physically possible (limiting) value of the parameter of nonlinearity of a structurally inhomogeneous medium as was reported by Zaitsev [12].

**EFFECT OF THE RESONANCE PROPERTIES OF MICROSCOPIC INCLUSIONS**

According to the results obtained in the previous section, the enhancement of the nonlinear properties of a microinhomogeneous medium is explained by an increase of local strain amplitudes at the soft inclusions relative to the small average strain in the material. In addition to the purely static effect, an extra increase of the local strain amplitudes is possible at the expense of the resonance excitation of the microscopic inclusions. Because of the compliance of the latter, they can have their own modes at sufficiently low frequencies with the wavelengths far exceeding their size. However, at frequencies close to resonance, it is impossible to characterize the nonlinearity of the medium by the static stress–strain relation as above, because the phase of the inclusion vibrations is shifted relative to the phase of the vibrations of the surrounding material. In this case, we can introduce some effective nonlinear parameters allowing us to compare the resonance nonlinear response (e.g., the levels at the combination frequencies) with the similar response at a quasistatic deformation. Because the resonance excitation of an oscillator with the quality factor $Q$ increases its vibration amplitude by a factor of $Q$ relative to the quasistatic response, one can expect that the results obtained above can be used in this case with the substitution $q \rightarrow qQ$, at least
for inclusions with equal resonance frequencies. However, such an approach is not always correct even in the case of identical inclusions, because, for the nonlinearly generated field components, the phase conditions of their generation play an important role. Hence, the effective parameter of nonlinearity of the medium may have different values for different components—for example, at the second harmonic and at the difference frequency. The variations of the nonlinear response in such resonance nonlinear effects was analyzed for gas bubbles in a liquid by Kobelev and Ostrovskii [3, 5] with an allowance for the distribution of the oscillators (bubbles) in their natural frequencies. Representing the results obtained in [3, 5] in terms of the quantities \( q \) and \( \nu \) introduced above, we obtain the expression for the coefficient of nonlinearity at the difference frequency \( \Omega = |\omega_1 - \omega_2| \) in the case of a biharmonic excitation:

\[
\Gamma^{(2)}_{\Omega} = P \nu v^2 Q_{\Omega},
\]

(15)

where \( P = 3\pi(3\gamma + 2)/4 \), \( \gamma \) is the adiabatic exponent of gas, \( q = K_1/K_2, K_1 \) and \( K_2 \) are the volume compressibilities of liquid and gas, and \( Q_{\Omega} \) is the quality factor of a gas bubble at the excitation frequency. Formula (15) was derived on the assumption that the resonance frequencies of all bubbles satisfy the condition \( \omega \sim \Omega_1, \Omega_2 \), and, in this particular case, the increase in the nonlinearity described by expression (15) corresponds to formula (11) with the substitution \( q \rightarrow q_Q \). If the bubble dimensions vary over a wider range, the coefficient \( \varepsilon_{\Omega} \) can also be represented by formula (15) with the quantity \( \nu \) being replaced by the relative number of the effectively “operating” bubbles \( \nu_{\text{eff}} = 1/Q_{\Omega} \). Then, the coefficient of nonlinearity will be increased by a factor of \( Q_{\Omega} \) rather than by a factor of \( Q_{\Omega}^2 \).

For the second harmonic, the effect of the bubbles resonant at the initial frequency is suppressed [3], because the contributions made by the bubbles with natural frequencies below and above the resonance have opposite phases. Thus, some increase in the coefficient of nonlinearity \( \varepsilon \) may occur only at the expense of the purely linear increase in the vibration amplitude of bubbles resonant at the frequency \( \Omega_2 \):

\[
\Gamma^{(2)}_{\Omega_2} = \tilde{P} \nu_{\text{eff}} q^2 Q_{\Omega},
\]

(16)

where \( \tilde{P} = \gamma \pi(4\gamma + 3)/298 \) and \( \nu_{\text{eff}} \) characterizes the relative number of bubbles resonant at the frequency \( \Omega_2 \):

\[
\nu_{\text{eff}} = \nu Q_{\Omega_2}.
\]

As a result, the coefficient of nonlinearity \( \varepsilon_{\Omega_2} \) expressed through the total bubble concentration takes the form

\[
\Gamma^{(2)}_{\Omega_2} = \tilde{P} \nu q^2 .
\]

(17)

which corresponds to the quasistatic case (11); i.e., the resonance effects cause no increase in the nonlinear response of the medium.

The examples considered above demonstrate that, only in a few special cases which are difficult to realize in practice, the effect of the resonance increase of the local strain amplitudes can be interpreted as an additional “softening” of the inclusions \( q \rightarrow q_Q \) and may cause a corresponding increase in the nonlinear parameters of the medium.

EFFECT OF THE SPATIAL ORIENTATION OF MICROSCOPIC INCLUSIONS

In the previous sections, we assumed that the inclusions isotropically respond to stress, which allowed us to consider a one-dimensional model of the medium. In real media, the properties of soft inclusions (e.g., cracks) can be strongly anisotropic. Such anisotropy changes the deformation characteristics of these inclusions and, hence, affects the nonlinear parameters of the medium. Evidently, the contribution of anisotropic inclusions to the nonlinear properties of the medium will be maximal if all inclusions have the spatial orientation corresponding to their maximal deformation under a given stress. Then, the deviations of the local stress—strain dependences from the linear will be maximal. Hence, to estimate the limiting value of the parameter of nonlinearity, we can use the results obtained in the previous sections by taking the value of the coefficient of elasticity of the inclusions corresponding to the defect orientation in the direction of maximal deformation. For other orientations, the estimate will be lower.

ESTIMATES OF THE LIMITING VALUE OF THE PARAMETER OF NONLINEARITY FOR SPECIFIC TYPES OF STRUCTURALLY INHOMOGENEOUS MEDIA

In this section, we discuss the limiting values for the parameter of nonlinearity in real media such as bubble—ous liquids and porous, granular, and cracked elastic media (these main types represent practically the entire spectrum of structurally inhomogeneous materials). We start with the conclusion derived as a result of our theoretical studies: the physically possible maximal value of the coefficient characterizing the increase in the parameter of nonlinearity of a structurally inhomogeneous medium with respect to the corresponding parameter of the homogeneous host material is determined by the ratio between the elastic moduli of the “stiff” and “compliant” components of the material under study.

As an example of a microinhomogeneous medium, we consider a liquid with gas bubbles. In such a medium, the quantity \( q \) characterizing the ratio between the coefficients of elasticity is equal to the ratio between the volume compressibilities of liquid \( K_1 \) and gas \( K_2 \) (\( q = K_1/K_2, K_2 = \gamma p_0 \), where \( p_0 \) is the gas pressure in the bubbles and \( \gamma \) is the adiabatic exponent). The volume compressibility of water, which is the most commonly encountered liquid, is \( K_1 \sim 2 \times 10^6 \text{ N/m}^2 \) under atmospheric pressure [16] and, for air at normal
conditions, \( K_q \sim 1.4 \times 10^8 \text{ N/m}^2 \) [16]. Hence, for water with gas bubbles, the maximal coefficient characterizing the increase in the parameter of the quadratic nonlinearity should reach \( q \sim 1.4 \times 10^4 \). Then, the maximal value of the parameter of the quadratic nonlinearity will be \( \Gamma^{\text{max}} \sim 10^4 \), which agrees well with experimental data [15].

For a porous medium, the coefficient characterizing the increase in nonlinearity is determined by the compliance of pores relative to that of the surrounding material. This relative compliance is determined by the ratio between the longitudinal and transverse Lame constants: \( q = \lambda/\mu \) [4]. In most elastic materials, this ratio makes several units, and, only in the so-called water-like materials, it may reach \( 10^4 \) [6]. Specifically, for porous plastoiles [6], the parameter of the quadratic nonlinearity may reach \( \Gamma \sim (3-8) \times 10^4 \) (at an optimal pore concentration).

Now, we estimate the maximal possible nonlinearity increase in a cracked medium. For this purpose, it is sufficient to estimate the minimal coefficient of elasticity of a single crack, which represents a soft inclusion in this case. For a crack in the form of a smooth cut of circular shape, the stress–strain relation can be expressed by the formula [17]

\[
\sigma = \frac{3\pi E}{16R(1-\eta^2)}d,
\]

where \( R \) is the transverse dimension of the crack, \( d \) is the distance between its edges, \( E \) and \( \eta \) are Young’s modulus and Poisson’s ratio of the elastic material, and \( \sigma \) is the stress. If some additional displacement \( \tilde{d} \) is superimposed on the initial displacement \( d_0 \) \( (d = d_0 + \tilde{d}) \), \( \sigma = \sigma_0 + \tilde{\sigma} \), formula (18) will relate the stress and displacement variations, \( \tilde{\sigma} \) and \( \tilde{d} \). After some transformations, relation (18) will take the form

\[
\tilde{\sigma} = \frac{3\pi E d_0}{16R(1-\eta^2)}\left(\frac{\tilde{d}}{d_0}\right) = \frac{E}{q} \varepsilon,
\]

where \( \varepsilon = \tilde{d}/d_0 \) is the relative deformation of the crack and \( E/\varepsilon \) is its effective elastic modulus. Comparison of expression (19) with the definition of the relative compliance of inclusions \( q \) shows that, according to (1), (3), and (4), the latter parameter will be determined by the formula

\[
q = \frac{16(1-\eta^2)(R)}{d_0^3}.
\]

For a cracked medium with the optimal concentration of microcracks, the maximal value of the parameter of quadratic nonlinearity will be of the same order of magnitude.

Finally, we consider one more class—the so-called contact-containing media including granular nonconsolidated or partially consolidated materials. In such media, the soft inclusions are represented by the contact regions between the grains, and the stiff component is the elastic grain material.

In this case, a typical inclusion will be a contact between two spherical surfaces with the curvature radius \( R \). The elastic force that acts in the contact region contracted by the length \( d \) under compression is determined by the Hertz formula [14]

\[
F = \frac{4ER^{1/2}}{3(1-\eta^2)}d^{3/2}.
\]

Introducing some initial contraction \( d_0 \) and its variation \( \tilde{d} \), we use formula (21) to derive an expression for the perturbation force \( \tilde{F} \):

\[
\tilde{F} = \frac{2ER^{1/2}}{(1-\eta^2)(R)}\left(\frac{d_0^3}{d_0^3}\right)\tilde{d}.
\]

The corresponding additional stress is \( \tilde{\sigma} = \tilde{F}/S = \tilde{F}/\pi a^2 \), where \( a \) is the radius of the contact region, and this radius is related to the curvature of the contact surfaces as \( a^2 \sim Rd_0 \) [14]. After simple transformations, the additional stress can be represented in the form

\[
\tilde{\sigma} = \frac{E}{\pi(1-\eta^2)}\left(\frac{d_0}{a}\right)^2 \varepsilon = \frac{E}{q} \varepsilon,
\]

where \( \varepsilon = \tilde{d}/d_0 \) is the relative deformation and \( q \) is the “relative compliance” of the contact (as in (19)). In the case under study, the quantity \( q \) can be expressed as

\[
q = \pi(1-\eta^2)\left(\frac{a}{d_0}\right)^2.
\]

From physical considerations, it follows that, for a smooth contact between two grains, the minimal possible value of \( d_0 \) is of the order of atomic dimensions \( d_0 \sim 10^{-4} \text{ mm} \), and the transverse dimension of the contact is about the size of a large dislocation loop \( a \sim 10-30 \text{ mm} \), as in the previous case. Thus, formula (24) yields the estimate of the compliance coefficient \( q \sim 10^4-10^6 \). Then, the maximal value of the parameter of the quadratic nonlinearity of a contact-containing medium may reach \( \Gamma \sim (q/4) \sim 10^4 \) for the optimal concentration of inclusions.
SUMMARY

The analysis performed in this paper shows that the model of a microinhomogeneous medium with identical isotropic inclusions allows one to estimate the maximal possible level of the elastic nonlinearity of the medium. This estimate is determined by the ratio between the linear elastic moduli of the host material and the soft inclusions rather than by the nonlinear properties of the inclusions (defects). Hence, there is a physical limit to the maximal level of the elastic nonlinearity of any microinhomogeneous medium, because of the limitation imposed on the ratio between the elastic moduli of the host material and the soft inclusions by the atomic structure of elastic bodies. Since this ratio can vary within four to five orders of magnitude, the limiting level, for example, of the quadratic parameter of nonlinearity makes $10^4 - 10^5$, which agrees well with the experimental data reported in the literature.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Fundamental Research, projects nos. 95-02-06411 and 98-02-17686.

REFERENCES


Translated by E.M. Golyamina