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Nonlinear Elastic Properties of Microinhomogeneous Hierarchically Structured Media

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Abstract—This paper examines the manner in which the internal structure of a medium can affect its nonlinear elastic properties. It is shown that, for the elasticity of a microinhomogeneous medium to possess a high level of nonlinearity, there must be a small proportion of a soft component present in the material. The way in which the hierarchical structure of the rigid and soft elements can influence the parameters of the medium is investigated. It is demonstrated that, if the rigid elements are arranged in a hierarchical structure, the internal levels present in the structure will not affect the further increase in the nonlinearity of the medium. On the contrary, if the soft elements are arranged in a hierarchical structure, then the nonlinearity of the medium can increase in a geometric progression. In real physical media, however, the range of variations in the elasticity of the components is limited to 3 or 4 orders of magnitude. Therefore, the ultimate increase in the nonlinearity coefficient compared to a homogeneous medium is likewise limited to the range stated.

INTRODUCTION

Recent years have seen an intensive proliferation of studies, both experimental and theoretical, into the nonlinear elastic properties of solids. Traditionally the origin of nonlinear elasticity has been ascribed to the anharmonism of interatomic interaction [1], and the nonlinear properties of isotropic media have been described by a five- or nine-constant theory of elasticity, in which quadratic or cubic corrections are applied to the linear Hooke's law [1]. In the important special case of planar deformation, under this approximation the stress–strain relation takes the form

$$\sigma = M\varepsilon(1 + \gamma\varepsilon + \beta\varepsilon^2 + \ldots), \qquad (1)$$

where M is the elastic modulus, $\gamma\varepsilon$, $\beta\varepsilon^2 \ll 1$, and γ and β are coefficients often called the quadratic and cubic nonlinear parameters. As found by experiments on several materials (glass, polymers, and some metals), γ and β are of the order of unity. This result is in good agreement with the view that the intermolecular potential obeys a weakly parabolic law. Hence, these materials can show a noticeable nonlinearity solely under a large amount of strain, $\varepsilon \sim 10^{-3}$ to 10^{-2}, close to their strength limit.

At the same time, numerous facts not fitting into the above view have been ascertained. Among other things, experiments have found that, for many kinds of terrestrial rock, polycrystalline metals, building materials, and porous, granular, and fissured media, the quadratic nonlinearity parameter is as high as 10^3–10^5, and its cubic counterpart, as high as 10^5–10^8 over a wide frequency interval extending from hertz to megahertz values [2]. It is likewise worth noting that the linear acoustic properties of these media have not exhibited an equally unusual behavior [3].

It has become clear, at least in principle, that the nonlinearity of such media is related in some ways to features in their microstructure (various defects, such as cracks, pores, and grains). Physical nonlinearity models based on consideration of the properties of such inclusions have been built for bubble-seeded liquids and porous elastic rubber-like media [3], granular materials [4], and fissured media [5].

Diverse as the above examples are, they have in common a structural feature that is responsible for the high linearity of such media. This is the presence of structural components sharply contrasting in linear elastic properties. The presence of soft defects, or inclusions, can give rise to locally high strains and stresses. For a sample as a whole, this implies an anomalously high nonlinearity under low average strains [4].

This mechanism of structure-conditioned nonlinearity is examined in detail by Zaitsev [6, 7] with reference to a generalized model of a microinhomogeneous medium. Noteworthily, the description proposed by Zaitsev [6] is in close agreement with the results obtained previously for specific cases [3–5]. However, the Zaitsev model [6] takes into account only one level of microstructural inhomogeneity: it assumes a two-component medium in which components sharply differing in compressibility are considered homogeneous. However, real microinhomogeneous media (such as terrestrial rock and polycrystalline materials) obviously have an inner structure (sometimes, of fractal character) with several hierarchical levels spanning between them a wide range of scales (see, for example, Nikolaev [8]). It is natural to expect that this kind of hierarchy can strongly affect the structure-conditioned local concentrations of strains and stresses and on the nonlinear properties of the medium. This matter requires a special analysis,

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which we will make, taking the generalized model of Zaitsev [6] as the point of departure, and consider, as an example, the specific case of a multicomponent granular medium.

A GENERALIZED MODEL OF A HIERARCHICALLY INHOMOGENEOUS MEDIUM

To begin with, consider the one-level approximation of structural inhomogeneity. Following Zaitsev [6], we choose a chain of point masses M joined together by elastic elements, or springs. For simplicity, their unperturbed lengths are assumed to be the same and equal to $L \ll \Lambda$, where Λ is the length of an acoustic wave (Fig. 1a). In modeling one structural inhomogeneity level of the medium, we suppose that the elasticity coefficient κ_1 of some springs is a small fraction of the elasticity coefficient κ of the remaining springs, such that $\kappa/\kappa_1 = Q \gg 1$. The linear number density of links in the chain is N, and the concentration of soft inclusions is N_1. Suppose, for convenience, that the chain has a cross sectional area of unity. Then, N and N_1 may be treated as volumetric concentrations in the medium, and the elastic force acting between the elements corresponds to the stress σ. To determine the nonlinear elastic properties of the medium, we suppose that the material of each spring deviates from the linear Hooke's law only slightly

$$\sigma = M\varepsilon[1 + f(\varepsilon)], \qquad (3)$$

$$\sigma = M_1\varepsilon[1 + f_1(\varepsilon)], \qquad (4)$$

where M and M_1 are the linear elastic moduli of spring material, such that, as assumed previously, $M/M_1 = \kappa/\kappa_1 \equiv Q \gg 1$, and the intrinsic nonlinearity of the material of the elastic elements, both rigid and soft, is small: $f(\varepsilon) \sim f_1(\varepsilon) \ll 1$.

We now find the elastic energy of springs as they are strained, taking advantage of the known relation of elastic energy to stress and strain, $\sigma = \partial W/\partial\varepsilon$, to arrive at an expression that relates the mean strain ε of the medium to the elastic stress σ

$$\sigma = \varepsilon\frac{M}{1+\zeta(Q-1)}\left\{1 + \frac{1-\zeta}{1+\zeta(Q-1)}f\left(\frac{\varepsilon}{1+\zeta(Q-1)}\right) + \frac{\zeta Q}{1+\zeta(Q-1)}f_1\left(\frac{Q\varepsilon}{1+\zeta(Q-1)}\right)\right\}. \qquad (5)$$

As equation (5) clearly shows, if the concentration of defects is low, $\zeta < Q^{-1}$, the argument of the function f_1 is $Q\varepsilon/[1 + \zeta(Q - 1)] \approx Q\varepsilon$. This implies that the strain suffered by the soft inclusions is by about a factor of $Q \gg 1$ greater than the mean strain ε of the medium. Because of this, the inclusions are shifted into a strongly nonlinear region under low mean strains and thus increase the nonlinearity of the medium as a whole.

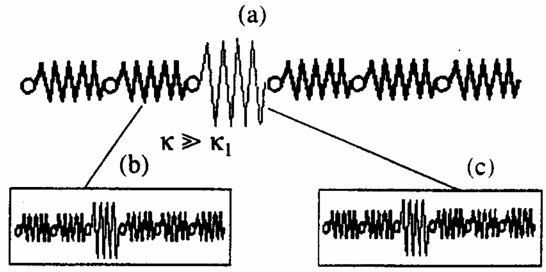

Fig. 1. Structure of a model microinhomogeneous medium: (a) a one-level approximation; (b and c) hierarchical structures of soft and rigid elements at the preceding level.

Let us analyze equation (5) in more detail for an instructive case where the nonlinearity obeys a power law, $f(\varepsilon) = f_1(\varepsilon) = \Gamma^{(n)}\varepsilon^{n-1}$. Here, $\Gamma^{(n)}$ is the dimensionless nonlinearity coefficient of the nth order, $n > 1$. Then, equation (5) takes the form

$$\sigma = \varepsilon BM\{1 + \varepsilon^{n-1}A\Gamma^{(n)}\}, \qquad (6)$$

where the coefficients A and B, which characterize the influence exerted by defects, are

$$B = [1 + \zeta(Q - 1)]^{-1}, \qquad (7)$$

$$A = \frac{1 + \zeta(Q^n - 1)}{[1 + \zeta(Q - 1)]^n}. \qquad (8)$$

According to equations (7) and (8), as long as the number density of defects is very low, such that $\zeta Q^n \ll 1$, both the linear elasticity of the medium and the nonlinearity coefficient are almost insensitive to their presence ($A \approx B \approx 1$). Further on, when $\zeta Q^n \gg 1$, but $\zeta Q \ll 1$, the linear elastic modulus still remains almost constant, $B \approx 1$, but the nonlinearity coefficient of the medium increases manifoldly ($A \approx \zeta Q^n \gg 1$). Upon a further increase in defect concentration ζ, when $\zeta Q \gg 1$, but as before $\zeta \ll 1$, the linear elasticity of the medium also decreases ($B \approx (\zeta Q)^{-1} \ll 1$). However, the nonlinearity coefficient of the medium remains anomalously high ($A \approx \zeta^{1-n} \gg 1$, because the exponent $n > 1$). Finally, with ζ tending to unity, the elastic properties of the chain are mainly determined by the elasticity of the soft inclusions ($B = Q^{-1}$; that is, $M_{mean} = M/Q$), and the nonlinearity coefficient reverts to its previous value ($A = 1$).

Thus, the nonlinearity coefficient $\Gamma^{(n)}$ passes through a maximum, which is achieved at a very low optimal concentration of inclusions

$$\zeta_{opt} = (n-1)^{-1}(Q-1)^{-1} - n(n-1)^{-1}(Q^n-1)^{-1} \approx (n-1)^{-1}Q^{-1} \ll 1. \qquad (9)$$

The nonlinearity growth factor is then given by

$$A(\zeta_{opt}) \approx (n-1)^{n-1}n^{-n}Q^{n-1} \gg 1, \qquad (10)$$

whereas the elastic modulus undergoes a far smaller change

$$B(\zeta_{opt}) \approx (n-1)/n \sim 1. \quad (11)$$

Qualitatively, the behavior of the dependences $A(\zeta)$ and $B(\zeta)$ is illustrated in Fig. 2.

To sum up, the simple one-level model of a microinhomogeneous medium examined above is an instructive illustration of the assertion made earlier about the role of soft defects, or inclusions: for the nonlinearity of a medium to rise strongly, the inclusions must be much softer than the surrounding material and their concentration must be low.

To take into account the hierarchical character of the structure, we now suppose that each element of the model examined above has an internal microinhomogeneity of its own, as shown in Fig. 1b and 1c. The influence of this new structural level can be analyzed by drawing upon the expressions derived for the one-level model.

Several points deserve special mention. First, by virtue of the mechanism examined above, the original intrinsic nonlinearity $f(\varepsilon)$ of rigid elements in equation (3) can strongly increase at the expense of their internal inhomogeneity shown in Fig. 1b. In particular, given the optimal defect concentration defined by equation (9), the quadratic nonlinearity parameter can, according to equation (10), increase by a factor of $Q/4$. However, as follows from equation (5), the inclusion of this element in the higher-scale structure shown in Fig. 1a will not change markedly the nonlinearity of the medium as a whole. The reason is that the coefficient of the rigid-element nonlinearity function f is of the order of unity, that is, $(1-\zeta)/(1+\zeta(Q-1)) \sim 1$.

The situation can be significantly different if the soft inclusions shown in Fig. 1a acquire an additional structural level (see Fig. 1c): their original nonlinearity can then increase by a factor of $Q/4$ at the expense of the smaller-scale structure. But, in contrast to the structuring of rigid elements, the nonlinearity of the higher-scale medium as a whole can increase again by a factor of $Q/4$, as follows from the form of the coefficient of the soft-element nonlinearity function f_1 in (5).

A similar reasoning applies to every additional hierarchical level in the structure of the medium. Thus, the multilevel arrangement of rigid elements will not cause an additional increase in the nonlinearity of the medium, but the multilevel structure of soft elements can cause the nonlinearity of the medium to increase in a geometric progression, ($\Gamma^{(2)} \sim (Q/4)^k$, where $k \geq 1$ is the number of hierarchical levels). In real physical media, however, the total range of component elasticity, Q_{max}, across all structural levels is limited to three or four orders of magnitude. Because of this, it usually makes sense to consider in real media at most three or four structural levels differing in elasticity, the contrast in elasticity between the components at each level

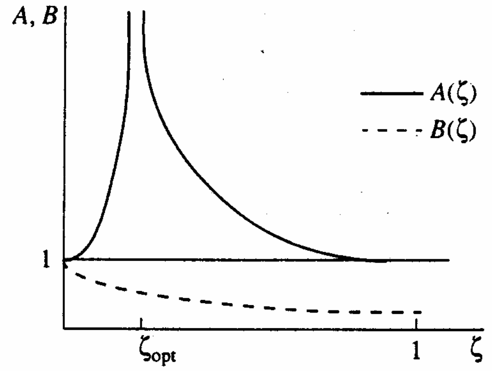

Fig. 2. Graph of the nonlinearity growth factor $A(\zeta)$ (full line) and the elastic modulus variation coefficient $B(\zeta)$ plotted against defect concentration.

being of the order $Q_i \sim Q_{max}^{1/k}$. Consequently, the resultant maximum increase in the nonlinearity coefficient compared to a homogeneous medium is of the same order in all cases, $\prod_{i=1}^{k} Q_i \sim (Q_{max}^{1/k})^k = Q_{max}$. That is, it is limited to Q_{max} all the same. This convincingly explains why the experimentally observable nonlinearity coefficient $\Gamma^{(2)}$ is not greater than 10^3 to 10^5 even for such multilevel media as terrestrial rock.

THE GENERALIZED MODEL AND CONTACT-CONTAINING STRUCTURES

As Zaitsev has shown [6, 7], the facts gleaned about the structural mechanism responsible for the existence of anomalous nonlinearity closely agree with what one can deduce from the specific models of nonlinear elasticity developed for bubble-seeded liquids [3], porous materials [9], and fissured media [5], provided the parameters Q and ζ introduced earlier are suitably chosen to fit those of the models listed.

Another important group of microinhomogeneous media is constituted by contact-containing materials, notably, granular media. In them, the material in the bulk of the grains performs the job of rigid elements, and the intergranular contacts play the role of soft inclusions.

Obviously, the structural inhomogeneity in such media can exist on several scales. For example, the pores between the grains can be filled by finer grains, which can in turn contain even smaller grains (a multifraction or multilevel structure of this kind is typical of many real unconsolidated media). It is of interest to see how this can affect the nonlinearity of the medium.

Let an aggregate of unit volume in a granular material be built up of spherical granules differing in size as follows. The largest particles, all of the same radius R_1, are packed at random. The voids between them are filled by particles of a radius $R_2 \ll R_1$, and so on. The voids are assumed to take up the same share of the total volume in each fraction (and be equal to $\alpha = 0.39$ for the random packing). Hence, the volumes taken up by the different size orders of particles have the following ratio:

$$V_1 : V_2 : \ldots V_n = 1 : \alpha : \ldots \alpha^n, \tag{12}$$

where n is the number of hierarchical levels of particles.

Let us write the material equation of the granular medium, allowing for its multilevel structure. Proceeding as in our previous development [4], the relation $\sigma(\varepsilon)$ may be written as

$$\sigma(\varepsilon) = \frac{\bar{n}E(1-\alpha)C_n}{3\pi(1-\nu^2)}\varepsilon^{3/2}, \tag{13}$$

$$C_n = \sum_{i=1}^{n} \alpha^{-(i-1)/2} = \frac{\alpha^{-n/2}-1}{\alpha^{-1/2}-1}. \tag{14}$$

Note that equation (13) holds for plane deformation as well [4]. Thus, equation (13) differs from the one-fraction case [4] in the factor C_n. This implies that additional lower-scale fractions appearing in the medium serve to increase its elasticity. To demonstrate, we expand equation (13) in a power series about some initial strain ε_0 and obtain for variations of stress an equation of the type (1) where

$$M = \frac{\bar{n}E(1-\alpha)B_n}{2\pi(1-\nu^2)}\varepsilon_0^{1/2}, \tag{15}$$

$$\gamma = 1/4\varepsilon_0, \tag{16}$$

$$\beta = -1/12\varepsilon_0^2. \tag{17}$$

It is thus seen that the nonlinear parameters expressed in terms of the initial compression of the medium remain unchanged as the packing hierarchy acquires additional levels. On the other hand, the initial compression ε_0 itself is determined by the initial static pressure applied to the medium. That is, given the same pressure, the initial compression will, according to (13) and (14), be lower in a multifraction medium, and the nonlinearity parameters of the medium will increase. In particular, it is an easy matter to show that, given a constant pressure, in a medium with n particle sizes, the quadratic nonlinearity coefficient $\gamma(\sigma = \text{const})$ increases in proportion to $(\alpha^{-1/2} - 1)^{2/3}(\alpha^{-n/2} - 1)^{-2/3}$. In a medium composed of five particle sizes, this will, for example, give a fourfold increase in nonlinearity.

It is likewise of interest to compare the results obtained for a granular medium with the finding of the previous section that the nonlinearity of a hierarchically arranged medium can increase manifoldly. To begin with, note that, as Zaitsev has shown [6], a one-fraction granular material can be viewed as corresponding to the model of a microinhomogeneous medium (equations (2)–(8), where the parameters ζ and Q are related to the characteristics of the granular medium $\zeta = \varepsilon_0$ and $Q = \varepsilon_0^{-3/2}$ such that $\zeta Q = \varepsilon_0^{-1/2} \gg 1$). This signifies two things. First, the contact concentration optimal for an increase in nonlinearity in such a medium has been exceeded. Second, the increase in the nonlinearity coefficient γ is determined, according to (8), by $A(\zeta Q \gg 1) \approx \zeta^{-1} = \varepsilon_0^{-1}$, which is significantly smaller than $A(\zeta_{opt} \approx Q^{-1}) \sim Q^{-1} \sim \varepsilon_0^{-3/2}$.

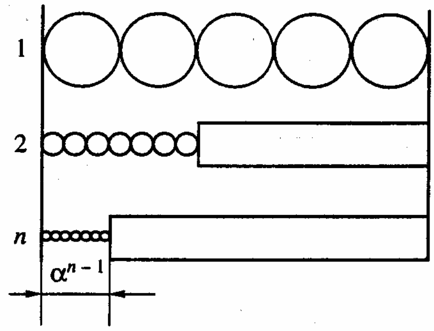

Fig. 3. Structure of a model multifraction granular medium.

Thus, the salient features of a granular medium are that the parameters Q and ζ, which characterize the elastic properties and concentration of inclusions, are not, by themselves, independent, and that the product ζQ occurs outside the region where a maximum rise in nonlinearity can be achieved. In terms of equivalent elastic elements, the above features will affect the hierarchical arrangement of the granular medium as follows. As the structure of equations (18)–(20) implies, a many-fraction granular medium can be modeled by a system of parallel-connected springs, as shown in Fig. 3. Each ith spring, of the same unit length $L = 1$, has a soft portion (corresponding to the soft intergranular contacts in the ith fraction) and a "perfectly rigid" portion of length $(1 - \alpha^n)$ (which, according to (12), corresponds to the volume of grains in all the preceding fractions).

When the fractions, or springs, are connected in parallel, the absolute changes in their lengths are the same. Therefore, for the ith fraction we have

$$(\varepsilon_0)_{i+1} = \alpha^{-i}(\varepsilon_0)_1, \tag{18}$$

$$(\varepsilon)_{i+1} = \alpha^{-i}(\varepsilon)_1, \tag{19}$$

where $(\varepsilon_0)_1 \equiv \varepsilon_0$ and $\varepsilon_1 \equiv \varepsilon$ represent the initial strain and its variations for the medium as a whole. Suppose that, in each spring shown in Fig. 3, its compressible portion is described by

$$\sigma = K\varepsilon^b, \tag{20}$$

where $K = \text{const}$, and the exponent in the case of Hertzian contacts is $b = 3/2$. Now, proceeding as in the case of equation (13) and using equations (18) and (19), we derive from equation (20) the following expressions for the linear elastic modulus and the nonlinearity coefficient:

$$M = Kb\varepsilon_0^{b-1}\alpha^{-bi}. \tag{21}$$

$$\gamma = \frac{(b-1)}{2\varepsilon_0}\frac{\alpha^i}{\alpha^i} \equiv \frac{(b-1)}{2\varepsilon_0}. \tag{22}$$

As can be seen, the rigid portion present in each ith element acts solely to change its linear elasticity (in compliance with equation (21)) and, in terms of the initial compression ε_0, does not affect the nonlinear properties, even if the soft portion takes up a small share of the total length. This finding contradicts, it would seem, what has been found in Section 1 of this paper, where it is argued that a soft inclusion appearing in a rigid medium should lead to a significant increase in its nonlinearity. It should be recalled, however, that the above finding holds when the relative softness Q and defect concentration ζ are specified independently of one another. In the granular-medium packing model in question, these parameters are, as has been noted, interrelated in such a way that the nonlinear properties turn out to be independent of the share taken up by the soft inclusions. Since the individual elastic elements, or particle-size fractions, are joined in parallel, the assertion that the nonlinear parameters are independent must hold for the entire medium as well.

On the other hand, our analysis has given insight into how a multicomponent granular medium must be arranged so that the structural mechanism responsible for an increase in nonlinearity discussed in the previous section can work. First, every fraction, or spring, made up of almost equally loaded grains of the same size, may contain extremely relieved contacts for which the effective softness and concentration are not subject to the constraint $\zeta Q = \varepsilon_0^{-1/2}$. In a truly one-dimensional medium, such "hanging" contacts cannot exist, but, in the bulk of real granular media, intergranular contacts can strongly differ in loading (because of, e.g., the nonspherical shape of the granules). Such soft inclusions can significantly increase the nonlinearity of the material [10, 11].

There is another possible way for the nonlinearity to increase, more closely related to the presence of a multiplicity of fractions. To demonstrate, equations (18) and (19) for the compression of parallel-connected fractions, or springs (see Fig. 3), are written for the case where the applied force causes the fractions to deform all at the same time. It may happen, however, that some of them do not begin to deform until a certain threshold stress is achieved. Then, such relieved fractions will, as follows from (16) and (17), possess a significantly higher nonlinearity and, as a consequence, will sharply enhance the nonlinearity of the medium as a whole. Such systems of contacts can be present in, e.g., real unconsolidated terrestrial rocks composed of differently sized and irregularly shaped grains. These systems are also exemplified by rough contacting surfaces, such as those of cracks in an elastic material, often having a fractal structure [12].

CONCLUSION

In summary, the examination reported herein has shown that, for a microinhomogeneous medium to have highly nonlinear elastic properties, there must be a small amount of a soft component present in the material. With only one structural level of such inhomogeneity, this can cause the quadratic nonlinearity parameter $\Gamma^{(2)}$ to increase by a factor of $Q/4 \gg 1$, where Q is the relative softness factor of inclusions. With the soft elements arranged in a hierarchical structure, the mechanism examined above can, in principle, cause the nonlinearity of the medium to increase in a geometric progression. In real physical media, however, the elasticity of the components varies over a limited range. Because of this, the resultant maximum increase in the nonlinearity coefficient compared to a homogeneous medium is likewise limited to three or four orders of magnitude.

A proper understanding of these general features of, and constraints on, the structural mechanism responsible for the increase in nonlinearity can make it easier to build physical models for the nonlinear elasticity of microinhomogeneous media.

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