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Experimental Study of Nonlinear Elastic Properties of Granular Media with Nonideal Packing

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Abstract – The manifestations of the so-called structure-induced nonlinearity have been experimentally studied in granular materials. The nonlinear acoustic effects can be explained by nonideal packing of granules. This results in inhomogeneous loading of the intergranular contacts and the presence of fully unloaded contacts, which is typical of the real granular media. The main differences in the behavior of the system from that expected for the model of ideal packing are discussed. Some estimates are obtained that demonstrate the effect of nonideal packing of granules.

In the earlier work [1], the nonlinear elastic properties of granular media were studied by analyzing the behavior of individual contacts between the granules. The granular medium was modeled by packing identical spherical particles in various modes, including random packing. The packing was ideal in the sense that the number of contacts between granules remained unchanged in the course of deformations, and the contacts, described by the Hertz theory, were assumed equally loaded. Within this model, the following stress–strain (σ–ε) relation was obtained for the plane deformation [1]:

$$\sigma = \frac{n(1-\alpha)E_s}{3\pi(1-\nu_s^2)}\varepsilon^{3/2}, \qquad (1)$$

where n is the average number of contacts per spherical particle, α is the packing porosity coefficient, and E_s, ν_s are the Young modulus and Poisson coefficient of the material.

For $n = 6$ and $\alpha = \pi/6$, equation (1) agrees with the earlier results [2, 3] for the simple cubic packing of spheres. In addition, nonlinear elastic properties of the granular medium were experimentally studied by harmonic generation [3]. To this end, a material was specially selected to provide for the best fit with the model of ideal packing. This was a fine lead shot comprising identical spheres with $R = 0.5$ mm, subjected to compression in a rigid metallic cylinder. This system corresponded to the case of plane deformation (lateral displacements forbidden).

Real granular media comprise, as a rule, granules having a distribution of sizes, with significantly nonspherical shapes. For this reason, it would be of interest to study the applicability of simplest models, assuming the ideal packing similar to those developed in [2, 3], to real systems. The results are important, e.g., for the application of the ideal theories to seismic diagnostics.

In this paper, the results of experimental investigation of nonlinear and linear elastic properties of granular media are considered from this last standpoint. The phenomena that do not conform to the model of ideal packing are established, the reasons for the discrepancies are discussed, and the effects of nonideal packing (variable number of intergranular contacts and differential loading) are estimated. The experimental setup and the procedure of measurements were similar to those described earlier [3].

MODEL CONSIDERATIONS

Before proceeding to a discussion of the results of measurements, we will briefly formulate the main conclusions of the model of ideal packing [1], which can be experimentally observed under properly selected conditions. According to equation (1), the layer of granular medium in the cylinder must behave like a nonlinear spring characterized by the following dependence of the elastic force on the displacement:

$$F = Qx^{\beta}\theta[x], \qquad (2)$$

where $\beta = 3/2$ is the Hertz nonlinearity index, $Q = Sn(1-\alpha)E_s/[3\pi(1-\nu^2)l^{\beta/2}]$, S is the cylinder cross-section area, l is the layer thickness, $\theta(x)$ is the Heaviside unit function, and x is the displacement measured from the upper boundary of a non-compressed layer. Once the oscillatory displacement of the plunger, $x = d_0 + d_\omega\cos\omega t$, ($d_\omega \ll d_0$), is specified, we can readily obtain the expressions for the force harmonics by expanding equation (2) into series in powers of the ratio of the oscillation amplitude to the static displacement ($\xi = d_\omega/d_0$):

$$F_\omega = Qd_0^\beta\beta\left[\xi + \frac{(\beta-1)(\beta-2)}{8}\xi^3\right]\cos\omega t, \qquad (3)$$

$$F_{2\omega} = Qd_0^\beta\frac{\beta(\beta-1)}{4}\xi^2\cos 2\omega t, \qquad (4)$$

$$F_{3\omega} = Qd_0^\beta\frac{\beta(\beta-1)(\beta-2)}{24}\xi^3\cos 3\omega t. \qquad (5)$$

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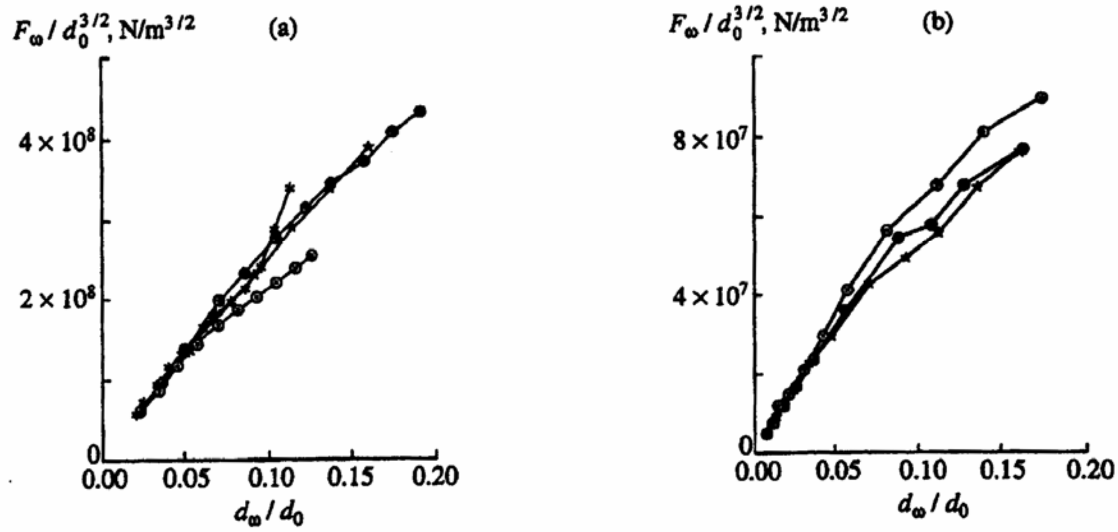

Fig. 1. Normalized first harmonic amplitude of the force ($F_\omega/d_0^{3/2}$) versus the normalized first harmonic amplitude (d_ω/d_0) of oscillations of the sample boundary: (a) lead shot, initial plunger shift (μm) d_0 = (★) 1.5, (⊕) 2.0, (⊙) 3.0, (∗) 4.0; (b) plastic granules, initial plunger shift (μm) d_0 = (★) 15, (⊕) 20, (⊙) 25.

These harmonics can be measured (as described in [3]) in order to determine the elastic modulus of the sample, $K = M(S/l)$ (where M is the longitudinal deformation modulus of the medium),

$$K = F_\omega/d_\omega, \quad (6)$$

and the quadratic and cubic nonlinearity parameters $\Gamma^{(2)} = \sigma''/\sigma'$, $\Gamma^{(3)} = \sigma'''/\sigma'$ [1]. Using (1) - (5), the latter parameters can be expressed as

$$\Gamma^{(2)} = (F_{2\omega}/F_\omega)4l/d_\omega, \quad (7)$$

$$\Gamma^{(3)} = (F_{3\omega}^2/F_\omega^2)24l/d_\omega. \quad (8)$$

According to equations (2) - (5), we can point out the following features of the stress–strain relationship (1) obtained assuming the ideal packing of granules: First, the square-root dependence of the elastic modulus on the initial displacement d_0, $K \sim d_0^{1/2}$. The character of this dependence is determined by the index of exponent in equations (1) and (2), which equals 3/2 for the Hertz type contacts, and is independent of the coefficient Q determined by the mode of packing, material properties, and experimental geometry. Secondly, equations (3) - (5) show that the relative magnitude of force harmonics is also determined only by the index β and the initial displacement d_0:

$$F_{2\omega}/F_\omega = (\beta - 1)4d_\omega/d_0, \quad (9)$$

$$F_{3\omega}/F_{2\omega} = (\beta - 2)6d_\omega/d_0. \quad (10)$$

Note that, because $\beta = 3/2$, the initial phase of the third harmonic is shifted by π, relative to the initial phases of the first and second harmonics. According to (4) and (5), the values of $F_{2\omega}$ and $F_{3\omega}$ are quadratic and cubic functions of the amplitude d_ω. According to (3), the elastic modulus must contain a small negative (i.e., tending to decrease) nonlinear correction, $K = K(\xi = 0)[1 - 1/32\xi^2]$; for normalized amplitudes $\xi < 0.25$ this correction does not exceed 0.2%.

The earlier experiments [3], performed with lead shot samples composed of identical small spheres, demonstrated a very close coincidence with the above-mentioned relationships between the elastic moduli and force harmonics and the displacements d_0 and d_ω. There was a good quantitative agreement between the measured values and the amplitudes of harmonics according to equations (9) and (10).

RESULTS AND DISCUSSION

The results presented below were obtained for a coarse lead shot (R = 1.65 - 1.8 mm) and plastic granules (R = 0.30 - 0.35 mm). These granules differed significantly from the fine shot used in [3] in respect to both nonspherical shape and a considerable scatter of grain sizes.

Figure 1 shows the normalized first harmonic amplitude of the force $F_\omega/d_0^{3/2}$ versus the dimensionless plunger displacement ξ. Different plots refer to various initial plunger shifts. According to the model (2), the coordinates employed in Fig. 1 must provide that all experimental points fit the universal curve $F_\omega/d_0^{3/2} = Q(3/2)[\xi - \xi^3/32]$. Moreover, no significant deviations from the linear dependence must be observed within the measurement accuracy. However, the experimental data show a noticeable scatter. The deviations from linearity are much greater than expected from (2) and (3) and have both negative and positive signs. An attempt to compare the experimental rigidity values to the estimate, based on equations (1) and (2) for the case of random packing ($n = 8.84$ and $\alpha = 0.39$ [4]), shows that the theoretical value is overstated by almost three times. Note that in the earlier experiment [3], the random-packing estimate coincided (within the measurement accuracy) with the experiential value.[1] Note that the porosity of lead shot was experimentally determined by comparing the weights of the dry and water-saturated samples; the values $\alpha = 0.4 - 0.42$ agree well with $\alpha = 0.39$ obtained in experiments [4] for a granular medium with randomly packed spheres.

The estimate of the average number of intergranular contacts calculated per granule by using equations (1) and (2) and the data of Fig. 1 yields $n = 3.5 - 4$. Note that even the least dense ideal packing mode (simple cubic) corresponds to $n = 6$. This estimate implies that a considerable proportion of intergranular contacts in real systems are partially or completely unloaded, so that the average number of "working" contacts (i.e., those determining the sample elasticity) can be significantly (by a factor of 2 - 3, as in our case) lower as compared to the random ($n = 8.84$) or closest ($n = 12$) packing modes. The same conclusion was derived from the data on the sound velocities measured in natural loose grounds. For example [5], in a ground with the porosity similar to the value typical of the closest packing, the sound velocity calculated assuming the simplest cubic packing showed a much better agreement with experiment.

Note also that the procedure of measurements involved a rearrangement of granules because each new value of the initial static displacement d_0 was obtained by completely pressing the sample with the plunger oscillating initially with the amplitude d_0. This procedure, monitored by oscillograph, was provided in order to eliminate gaps between the plunger and the sample (as indicated by the absence of "cuts" in the oscillogramm). The repacking seems to account for the difference in the shapes of the curves observed in Fig. 1 for various d_0. The increase and decrease in the sample rigidity with increasing amplitude of oscillations ξ seem to reflect various types of gradual "switching on and off" of weakly loaded contacts in various packings. The experimental data show that the rigidity K_0 increases with the initial displacement d_0 more rapidly than it might be expected according to the square-root law following from equations (1) and (2). This behavior is also due to the switching of additional contacts at higher loads. The result of the power regression according to $K \sim d_0^\gamma$ yields $\gamma = 0.54$ and 0.69 for the lead shot and plastic granules, respectively.

[1] Erratum: There is a misprint in the data presented in Table 1 of [3]. The values of power indices must be greater by one order of magnitude.

Now we will turn to an analysis of the amplitude dependence of the force harmonics. Figure 2 shows the second harmonic amplitude $F_{2\omega}$ of the force versus the plunger amplitude. The regression procedure for the exponent in $F_{2\omega} \sim d_\omega^\gamma$ yields $\gamma \cong 1.8 - 2.3$ for various initial displacements. The dependence is close to the quadratic law expected for the ideal packing. However, the deviation from $F_{2\omega} = 2$ is not merely due to the measurement error, but reflects the different properties of the ideal and real packing of granules. It should be recalled that the model of ideal packing with constant number of contacts also predicts a definite ratio of the force harmonics. The calculation of the absolute values of the harmonics would require the knowledge of the coefficient Q entering in equation (2), which is determined by the properties of the material and the mode of packing. In contrast to the earlier experiment [3], where the ratio of harmonics agreed well with the value following from equation (9), the level of the second harmonic amplitude obtained in this work (Fig. 2) exceeds the theoretical values by 20 - 50%.

Now we will consider the behavior of the third harmonic. Figure 3 shows the amplitude $F_{3\omega}$ versus the plunger displacement amplitude d_ω. According to equation (4), based on the model of ideal packing, the dependence must be cubic and this was really observed in the earlier experiment [3]. The power regression, according to $F_{3\omega} \sim d_\omega^\gamma$ performed for the data of Fig. 3a, gives a curve that is close to the quadratic dependence, rather than to the cubic one ($\gamma = 1.8 - 2.0$ for various initial displacements). The data for the plastic granules presented in Fig. 3b exhibit a transition from a nearly cubic dependence at small oscillation amplitudes to an almost quadratic law at large amplitudes.

Our experiments did not offer the possibility of determining the initial phase of the third harmonic, which, according to the model of ideal packing (5), must have a value of π. On the other hand, if the third harmonic is related to the "clapping" nonlinearity of weakly loaded contacts (completely unloaded during a certain part of the plunger oscillation period), then the initial phases of the third and first harmonics must coincide. Thus, assuming that the increase in the plunger amplitude d_ω leads to a contribution from "clapping" contacts in addition to the permanently loaded contacts, and taking into account the opposite phases of these contributions, we may expect that the amplitude $F_{3\omega}$ would grow at a lower rate as compared to the cubic law (5). This conclusion agrees with the data of Fig. 3. Moreover, the competition of the two types of nonlinearity responsible for the third harmonic may, in principle, lead to the appearance of local minima of $F_{3\omega}$ at certain values of the plunger oscillation amplitude. Such minima were observed in experiments for some particular packings of both lead and plastic granules.

Note that, in addition to the above-mentioned qualitative features, we can also expect that the presence of

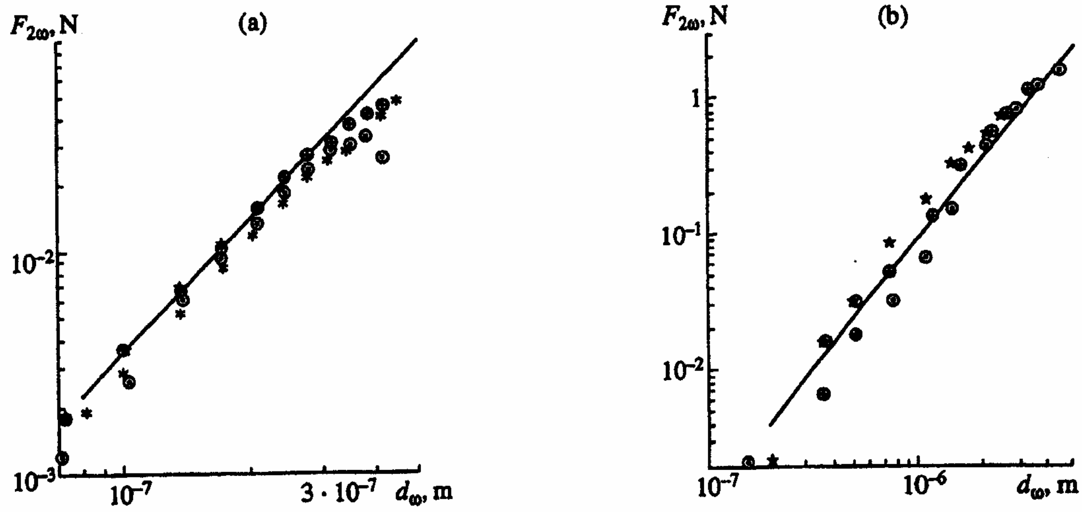

Fig. 2. Double logarithmic plot of the second harmonic amplitude of the force ($F_{2\omega}$) versus the first harmonic amplitude (d_ω) of oscillations of the sample boundary (the slope of the curves corresponds to the quadratic dependence $F_{2\omega} \sim d_\omega^2$): (a) lead shot, initial plunger shift (μm) d_0 = (★) 1.5, (⊕) 2.0, (⊙) 3.0, (*) 4.0; (b) plastic granules, initial plunger shift (μm) d_0 = (★) 15, (⊕) 20, (⊙) 25.

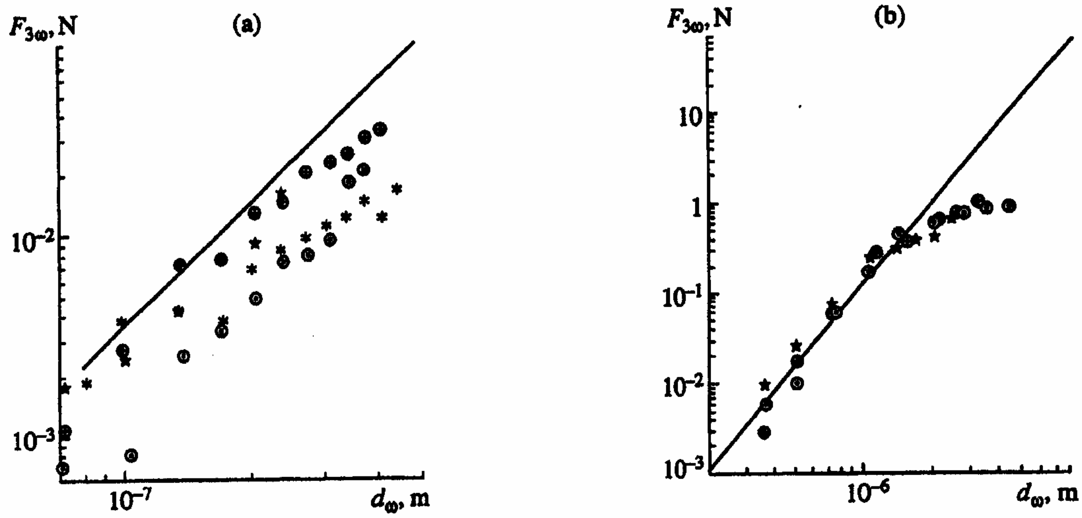

Fig. 3. Double logarithmic plot of the third harmonic amplitude of the force ($F_{3\omega}$) versus the first harmonic amplitude (d_ω) of oscillations of the sample boundary: (a) lead shot, initial plunger shift (μm) d_0 = (★) 1.5, (⊕) 2.0, (⊙) 3.0, (*) 4.0 (the slope of the curve corresponds to the quadratic dependence $F_{2\omega} \sim d_\omega^2$); (b) plastic granules, initial plunger shift (μm) d_0 = (★) 15, (⊕) 20, (⊙) 25 (the slope of the curve corresponds to the cubic dependence $F_{3\omega} \sim d_\omega^3$).

weakly loaded contacts would affect the magnitude of $F_{3\omega}$ as compared to the level anticipated for the model of ideal packing:

$$F_{3\omega}/F_{2\omega} = 1/12\xi. \tag{11}$$

(as was observed in the earlier experiments [3]). Indeed, not only the data of Fig. 3 exhibit qualitatively different behavior as compared to that expected on the basis of equations (5) and (11), but the absolute values of the $F_{3\omega}$ are higher by more than an order of magnitude than the values predicted by these formulas. Now we will make some estimates, which will demonstrate that this fact also agrees with the proposed effects of the nonideal packing of granules. For the simplicity, the consideration is restricted to two types of intergranular contacts. We will assume that, in addition to the main fraction of contacts loaded by the plunger displacement d_0, there exist a fraction ν (for this estimate, ν < 1) of weakly loaded contacts, corresponding to a displacement μd_0 with $\mu < 1$. When harmonic oscillations with the amplitude $\xi = d_\omega / d_0$ are excited on the boundary of this sample, we have, by analogy with (2):

$$F = Q_0 d_0^{3/2}\,[1 + \xi\cos\omega t]^{3/2}\theta\,[1 - \xi\cos\omega t]$$
$$+ \nu\mu^{3/2}\,(Q_0 d_0^{3/2})\left[1 + \frac{\xi}{\mu}\cos\omega t\right]^{3/2}\theta\,[\mu - \xi\cos\omega t]. \quad (12)$$

The consideration will be restricted to small amplitudes $\xi < \mu$, when the "clapping" nonlinearities can be ignored. On expanding (12) into series in powers of ξ, we readily obtain that the second term of (12) gives the corrections to the amplitudes of force harmonics that have the orders of $\nu\mu^{1/2}$, $\nu\mu^{-1/2}/4$, and $\nu\mu^{-3/2}/24$ for the first, second, and third harmonics, respectively. Therefore, the relative value of the correction is lower by a factor of μ^{-1} for each next harmonic number. For example, with $\mu \sim 10^{-2}$, the correction due to the second term in equation (12) does not exceed 10% for the first harmonic, but increases to 100 - 200% for the second harmonic, and reaches up to 20 - 40 times the initial value for the third harmonic. These estimates agree quite well with the observed deviations of the force harmonic amplitudes from the predictions based on the model of ideal packing.

CONCLUSION

Experiments on the harmonic generation in granular media revealed the following discrepancies related to nonideal packing of granules (i.e., to the presence of weakly loaded contacts):

(1) The data on the first (main) harmonic of the force exhibit a much lower elasticity of the sample and a more rapid increase in the elasticity with the static displacement as compared to the level expected for the ideal random packing.

(2) The second harmonic amplitude shows an increase by 50 - 100% as compared to the ideal model and deviates from the strict quadratic amplitude dependence; the discrepancies cannot be explained by experimental errors.

(3) The third harmonic has the amplitude exceeding the ideal level by a factor of 10 - 30, shows a change in the sign of the phase, and exhibits quadratic rather than cubic amplitude dependence.

All these features disagree with conclusions derived from the model of ideal packing. However, they can be quite naturally interpreted (both qualitatively and by the order of magnitude) assuming a nonideal packing of granules.

A more detailed quantitative analysis of conclusions following from the refined model is in progress.

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